Densest/Heaviest k-subgraph on Interval Graphs, Chordal Graphs and Planar Graphs

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Problem Definition:

Densest k-Subgraph Problem(DS-k):

- Input:$G(V, E)$, $k > 0$.
- Output: an induced subgraph D s.t. $|V(D)| = k$.
- Goal:Maximize $|E(D)|$.

Heaviest k-Subgraph Problem(HS-k):

- Input:$G(V, E)$,$w : E \to R^+$,$k > 0$.
- Output: a induced subgraph D s.t. $|V(D)| = k$.
- Goal:Maximize $\sum_{e \in E(D)} w(e)$.

Previous Results

- NP-hard even on chordal graphs(Corneil,Perl.1984) and planar graphs(Keil,Brecht.1991).
- n^δ-approximation for some $\delta < 1/3$ (Feige,Kortsarz,Peleg.2001).
- n/k-approximation(Srivastav,Wolf.1998;Goemans,1999).
- No PTAS in general(Khot.2004).
- PTAS for dense graph(Arora, Karger, Karpinski.1995).

Previous Results

- Some better approximation for special k.
- HS-k is in **P** on trees(Maffioli.1991), co-graphs(Corneil,Perl.1984) and chordal graph if its clique graph is a path(Liazi,Milis,Zissimopoulos.2004).
- PTAS on chordal graph if its clique graph is a star(Liazi,Milis,Pascual,Zissimopoulos.2006).
- OPEN: complexity on interval graphs(even for proper interval graphs).

Our Results

- Proper interval graphs(unknown): PTAS.
- Chordal graphs(NP-hard): Constant approximation.
- Planar graphs(NP-hard): PTAS.

Proper Interval Graphs-A simple 3-approximation

Densest disjoint clique k-subgraph(DDCS-k) problem:
Find a (not necessarily induced) subgraph $G'(V', E')$ such that

- $|V'| = k$;
- G' is composed with several vertex disjoint cliques;
- $|E'|$ is maximized.

Proper Interval Graphs-A simple 3-approximation

DDCS-k can be solved by Dynamic Programming: Let $DS(i,l)$ be the optimal solution of DDCS-l problem on $G(V_{1\dots i})$.

$$DS(i,l) = \max_{(j,x)\in\mathcal{A}}\{DS(j,x) + \binom{l-x}{2}\}$$

where $\mathcal{A}$ is the feasible integer solution set of the following constraints system:
$1 \leq j < i, 0 \leq x \leq l, l-x \leq i-j, l-x \leq i-q_G(i)+1$.

Proper Interval Graphs-A simple 3-approximation

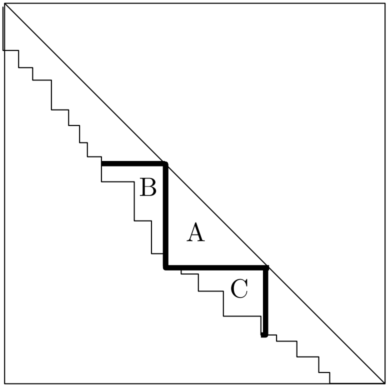
An optimal DDCS-k solution is a 3-approximation of DS-k problem.
We construct a DDCS-k solution OPT_{DDCS} from an optimal solution OPT_{DS} of the DS-k problem such that $|OPT_{DDCS}| \geq 1/3 \cdot |OPT_D S|$.

Proper Interval Graphs-A simple 3-approximation

Construction(Greedy):

- Repeatedly remove the vertices and all adjacent edges of a maximum clique from OPT_{DS}.
- Take OPT_{DDCS} as the union of these maximum cliques.

Proper Interval Graphs-A simple 3-approximation



Proper Interval Graphs-PTAS

Def: *overlap number* $\kappa_G(v)$ as the number of maximal cliques in G containing v.

The h*-overlap clique subgraph* H is a subgraph of G such that $\kappa_H(v) \leq h$ for all $v \in V(H)$.

For example, a disjoint clique subgraph is a 1-overlap clique subgraph.

Proper Interval Graphs-PTAS

densest h-overlap clique k-subgraph(DOCS-(h,k)) problem:
Find a (not necessarily induced) subgraph $G'(V', E')$ such that

- $|V'| = k$;
- G' is a h-overlap clique subgraph of G;
- $|E'|$ is maximized.

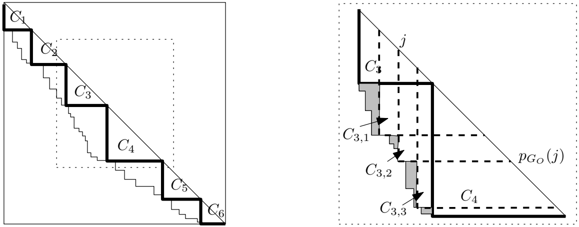
DOCS-(h,k) can be also solved by dynamic problem if h is a constant.

Proper Interval Graphs-PTAS

Similarly, we construct a DOCS-(h, k) solution OPT_{DOCS} from an optimal solution OPT_{DS} of the DS-k problem such that $|OPT_{DOCS}| \geq (1 - \frac{4}{h/2-1}) \cdot |OPT_{DS}|$.

So, in order to get a $1 - \epsilon$ approximation, it is enough to set $h = 2 + 8/\epsilon$.

Proper Interval Graphs-PTAS



(a) (b)

Chordal Graph

A graph is *chordal* if it does not contain an induced cycle of length k for $k \geq 4$.

A *perfect elimination order* of a graph is an ordering of the vertices such that $Pred(v)$ forms a clique for every vertex v, where $Pred(v)$ is the set of vertices adjacent to v and preceding v in the order.

Thm: A graph is chordal if and only if it has a perfect elimination order.

Chordal Graph

Maximum Density Subgraph Problem(MDSP)

- Input $G(V, E)$, vertex weight $w : v \to \mathbb{R}^+$,
- Output: an induced subgraph $G'(V', E')$.
- Goal: maximize the density $\frac{(\sum_{v \in V'} w(v) + |E'|)}{|V'|}$.

This problem can be solved optimally in polynomial time by reducing to a parametric flow problem [Gallo,Grigoriadis,Tarjan.1989].
Important Fact: $w(v) + d_{G'}(v) \geq \rho$.

Chordal Graph

The high level idea :
We run the above MSDP algorithm on our given graph with $w(v) = 0$ for all $v \in V$

- we get a subgraph G' of size k, we have exactly an optimal solution for the DS-k problem.
- If we get a smaller subgraph, we repeat the MSDP algorithm in the remaining graph and add the solution in.
- If we we get a larger subgraph, we need to pick some vertices in this subgraph to satisfy the cardinality constraint without losing much density.

Chordal Graph

Densest-k-Subgraph-Chordal(G(V,E))

1: $V_0 = \emptyset$; $i = 0$;
2: $i = i + 1$;run MSDP in the remaining graph $G(V - V_{i-1}, E(V - V_{i-1}), w_i(v) = d(v, V_{i-1})$. let the optimal subgraph(subset of vertices) be V_i' and the density be ρ_i.
3: **if** $|V_{i-1}| + |V_i'| < k/2$ **then**
4: $\quad V_i = V_{i-1} \cup V_i'$ and go back to step 2.
5: **else if** $k/2 \leq |V_{i-1}| + |V_i'| \leq k$ **then**
6: $\quad V_i = V_{i-1} \cup V'$.
7: **else if** $|V_{i-1}| + |V_i'| > k$ **then**
8: $\quad V'' = Pick(V_i', w_i)$ and $V_i = V_{i-1} \cup V''$;
9: **end if**
10: Arbitrary take $k - |V_i|$ remaining vertices into V_i.

Chordal Graph

Pick(V_t',w)

Compute a perfect elimination order for V', say $\{v_1, v_2, \dots, v_m\}, m > k/2$. $V'' = \emptyset$.
for i=m to 1 **do**
  if $|Pred_{V_t'}(v_i)| \geq \rho_t/2$ **then**
    if $|V''| + |Pred_{V_t'}(v_i)| + 1 \leq k/4$ **then**
      $V'' = V'' \cup \{v_i\} \cup Pred_{V_t'}(v_i)$.
    else if $k/4 < |V''| + |Pred_{V_t'}(v_i)| + 1 \leq k/2$ **then**
      $V'' = V'' \cup \{v_i\} \cup Pred_{V_t'}(v_i)$; return V''.
    else if $|V''| + |Pred_{V_t'}(v_i)| + 1 > k/2$ **then**
      Add into V'' v_i and arbitrary its $k/2 - |V''| - 1$ predecessors; return V''.
    end if
  else if $w(v_i) + |Succ_{V_t'}(v_i)| > \rho_t/2$ **then**
    $V'' = V'' \cup \{v_i\}$; If $|V''| > k/4$ return V'';
  end if
end for

Chordal Graph

Suppose $OPT = G^*(V^*, E^*)$.

If $|E(SOL \cap V^*)| \geq |E^*|/2$, then the algorithm is a 1/2-approximation.

If not.....

Chordal Graph

Analysis Sketch:

Let $I_i = V_i \cap V^*$ and $R_i = V^* \setminus I_i$. Since we get the the optimal solution V_i' on MDSP instance $G(V - V_{i-1}, E(V - V_{i-1}), w_i(v) = d(v, V_{i-1})$ at step i, we have

$$\begin{aligned}\rho_i &= \tfrac{|E(V_i')|+d(V_{i-1},V_i')}{|V_i'|} \geq \tfrac{|E(R_{i-1})|+d(V_{i-1},R_{i-1})}{|R_{i-1}|} \geq \tfrac{|E(R_{i-1})|+d(I_{i-1},R_{i-1})}{|R_{i-1}|} \\ &\geq \tfrac{|E(R_{i-1})|+d(I_{i-1},R_{i-1})}{k} = \tfrac{|E^*|-|E(I_{i-1})|}{k} \geq \tfrac{|E^*|-|E(I_t)|}{k} \geq \tfrac{|E^*|}{2k} = \tfrac{\rho^*}{2}.\end{aligned}$$

for all $i \leq t$.

Chordal Graph

we can prove $\rho_i \geq \rho_{i+1}$ for all i.
We can also prove if $\rho_i > k/4$ then $i = 1$.
If $\rho_t > k/4$, and recall $d_{V_1}(v_1) \geq \rho_1 > k/4$, So, a clique of size at least $k/4$, a 16-approximation.
If not,...

Chordal Graph

In **Pick**:
we can see
$w(v) + d_{V''}(v) = w(v) + |Pred_{V''}(v)| + |Succ_{V''}(v)| \geq \rho_t/2$.

So

$$\begin{aligned} \rho'_t &= \frac{E(V'')+d(V'',V_{i-1})}{|V''|} = \frac{1/2\sum_{v\in V''} d_{V''}(v)+\sum_{v\in V''} d(v,V_{i-1})}{|V''|} \\ &\geq \frac{\sum_{v\in V''} d_{V''}(v)+\sum_{v\in V''} w_i(v)}{2|V''|} \geq \frac{\rho_t}{4}. \end{aligned}$$

Planar Graph

Sketch:

- Decompose the planar graph into a series of K-outerplanar graphs.
- Solve the problem in each outerplanar graph.
- Recombine the solution.

Standard Baker's technique, but some more details...omit here.

Thank You!

- thanks to Jian XIA and Yan ZHANG for discussions on proper interval graphs.