Incentivizing Truth Exploration and Honest Reporting: A Contract Design Approach

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ABSTRACT

In this paper, we study a Principal-Agent problem in which a principal incentivizes an agent by establishing a payment contract that encourages the agent to exert costly effort in exploring the true state of the environment, which is of interest to the principal, and then report the findings. We consider two feedback setups: (1) the true state is ultimately observable by the principal, and (2) only some noisy feedback related to the true state is observable. In the first setup, we demonstrate that the optimal contract is the one that pays the agent only when the report matches the ground truth, and we derive an efficient algorithm to compute this optimal contract. In the second setup, we design a BDD contract and show its approximate optimality with respect to the optimal honest-reporting incentivizing contract, both theoretically and empirically. Furthermore, we introduce a sufficient condition under which the optimal contract encourages honest reporting.

KEYWORDS

Contract Theory; Stackelberg Game; Information Acquisition

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1 INTRODUCTION

In real life, delegating the task of truth investigation is common. In business consulting, business owners may hire firms like McKinsey or Boston Consulting Group to conduct market research and develop strategic plans for them. In human resource management, companies lacking expertise in recruitment or performance evaluation may engage HR management firms to evaluate their candidates and employees. These scenarios can be modeled as a **Principal-Agent** interaction problem, where the principal values certain information that she cannot access directly due to a lack of expertise. She seeks assistance from an agent capable of uncovering this information. The difficulty she faces when implementing this delegation is that the information acquisition process typically remains opaque to her, which may result in insufficient incentives for the agent to conduct diligent research (known as the *moral hazard* issue, [16]). Since the truth investigation can be quite costly for the agent, such as requiring detailed industry surveys, he may choose to reduce his level of effort and be strategic about what to report, e.g., merely providing superficial, unsubstantiated guesses while claiming they were obtained from thorough work.

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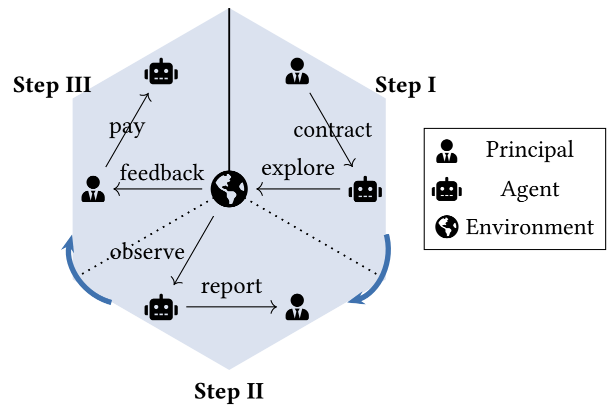

Figure 1: The Principal-Agent interaction process considered in this paper.

In this paper, we investigate the problem of incentivizing the agent to exert effort in the truth exploration process and to make a valid report from the perspective of **contract design**, following a recent trend of using contract theory to address various particular Principal-Agent problems [1, 13, 23]. Specifically, we consider a Stackelberg game [3] where the principal announces a policy of payment to the agent in advance, which specifies the exact payment she delivers to him given his actual report and some feedback that is closely related to the ground truth. The agent then selects his strategy of effort investment and reporting to maximize his net utility. We summarize the complete interaction procedure as follows (See Figure 1 for an illustration):

- **Step I.** The principal designs a contract and presents it to the agent. The agent then chooses the level of effort and begins exploring the truth.
- **Step II.** The agent observes information about the environment and then determines what to report to the principal.
- **Step III.** The principal receives feedback from the environment and then pays the agent according to the contract.

We present a real-world scenario that our model can capture in the following example.

Example 1 (Delegated Market Research). Financial markets change rapidly, and only a few professionals can accurately access internal market information, which is valuable for making trading decisions. An investor (Principal) may hire a financial expert (Agent) to investigate market dynamics. The expert conducts research and subsequently reports the findings, which the investor can use as a reference for investment decisions. The investment outcome can be characterized by public information, such as stock prices (Feedback). The investor can utilize this public information, or even the market state at the time of decision-making if available, to assess the accuracy of the expert's report and adjust payment accordingly.

In our model, the ground truth is represented by an environment state $\theta \in \Theta$, where Θ is the set of all possible environment states. We assume the agent's exploration process may not always succeed. If successful, the agent observes the correct θ; if not, no information is gained. Different levels of agent effort correlate with varying success rates and costs, with the reasonable assumption that higher success probabilities incur greater costs for the agent. Regardless of whether the exploration is successful, the form of the agent's report is an element of Θ representing his belief about the truth.

In this paper, we analyze the problem from the principal's perspective, assuming both players are utility maximizers. Our focus is on designing an optimal contract to maximize the principal's utility while considering additional factors. For example, a desirable property of a contract is that it incentivizes the agent to report findings honestly. In our model, when exploration is successful, the strategic agent is aware of the payment associated with each reporting choice and may opt for the one that maximizes payment, regardless of its truthfulness. Thus, when evaluating a specific contract, we assess its ability to promote truth-telling, as accurate reporting is typically most beneficial to the principal, while the mutual trust built by honesty increases the likelihood of long-term cooperation between the principal and agent. Consequently, the ideal contract is not only a payment policy that optimizes the principal's utility but also serves as a mechanism that encourages the agent's honesty.

1.1 Our Contributions

In this research, we explore two distinct models that differ primarily in the type of feedback received from the environment. Accordingly, our contributions are twofold:

- In the model where the true state of the environment is ultimately usable by the principal to evaluate the agent's report, we prove that the optimal contract pays only when the report matches the ground truth. Based on this, we present an algorithm that efficiently computes the optimal contract by solving a polynomial number of linear programs.
- In the model where the principal can only observe some related feedback instead of the ground truth, we design a BDD contract and prove its approximate optimality relative to the ideal truth-telling incentivizing contract, both theoretically and empirically. Furthermore, we prove that if a clear distinction exists between the benefits to the principal from truthful versus non-truthful reports, an optimal contract can be constructed to encourage honest reporting.

2 RELATED WORK

2.1 Contract Theory

Contract Theory is a traditional branch of economics, dating back to at least 1979 [14, 19]. In recent years, a series of studies has emerged that studies contracts through the lens of the theory of computation, led by [10]. The authors of [10] raise concerns regarding the complexity and unintuitiveness of optimal contracts, while exploring the approximation ratios of simpler contract forms, such as linear contracts. A line of research [15–17] on the menu of contracts is particularly relevant to our work, as the agent reporting process in our model is similar to the agent's selection of a contract from a proposed menu in their models. The difference, for example, is that in these models, the agent knows his private type in advance and does not need to exert effort to explore the environmental information, and the menu is constructed to either accommodate various types of agents for greater benefits [15, 16] or to assist in learning the private information [17]. There are other papers studying combinatorial contracts from a computational perspective [8, 9, 11, 12], but they are less related to our topic. Recently, an increasing number of works have focused on extending contract theory to other research areas. For instance, [13] studies the delegation problem of sequential probing, while [1, 23] consider contracts for machine learning tasks.

2.2 Proper Scoring Rules

Scoring rules are payment policies designed to incentivize risk-neutral experts to provide their probability assessment for an uncertain event [5]. A scoring rule is proper if the expert's optimal strategy, under any belief he might possess, is to report that belief [21]. Some recent studies [6, 18, 20] examine how to incentivize agents, using proper scoring rules, to access a costly signal that refines their beliefs. The setups of these papers differ from ours, for example, in the following aspects: (1) The form of the reports: in our model, the agent submits only a prediction of the true state, whereas scoring rules typically require a believed probability distribution over all possibilities. (2) The optimization problems: in our model, the objective is principal utility maximization, while these papers have different goals. For instance, some aim to maximize the agent's additional benefit when accessing the signal, treating the principal's preferences as budget constraints [6, 20]. The authors of [22] establish a connection between the menu of contracts and proper scoring rules within a classic hidden-action principal-agent model, featuring the novel aspect that the agent can choose to observe a costly signal correlated with the outcome. They investigate the problem of incentivizing signal acquisition at minimal cost.

3 CONTRACTING ON TRUE ENVIRONMENT STATES

3.1 The Model

We introduce the Principal-Agent model used in this section. In this model, there is a variable $\theta \in \Theta$ representing the true state of the environment. We assume Θ, the set of all possible states, to be discrete and let $m = |\Theta|$, where $|\cdot|$ denotes the size of a finite set. The state θ is randomly sampled by the Nature from a prior distribution $\mathbb{P}_\theta$. We let P_θ be the probability that the true state is realized to be θ and define $\underline{P} = \min_{\theta\in\Theta} P_\theta$. Two players interact in this environment: A **principal** (referred to as she/her) wants to uncover the true state θ and seeks to delegate this exploration task to an **agent** (referred to as he/him) with expertise. They both know the prior distribution $\mathbb{P}_\theta$. The agent's actions have two stages. In the first stage, he chooses a proper level of his exploration effort. Formally, say the agent has n actions. Each action $a \in [n]$ (throughout this paper, we let $[N]$ be the set $\{1, 2, ..., N\}$ for any $N \in \mathbb{N}^+$) is associated with a cost $c_a \geq 0$ and a success rate $q_a \in [0, 1]$ such that, with probability q_a, he successfully identifies the true state θ, while with probability $1 - q_a$, he gains no additional information about it. Without loss of generality, we assume that

(1) The more effort the agent puts in, the more likely the investigation is successful: $c_a < c_{a'} \iff q_a < q_{a'}, \forall a, a'$.
(2) The agent can invest no effort: $c_1 = 0$ and $q_1 = 0$.

In the second stage of the agent's actions, he reports his result $\hat{\theta}$, representing his prediction of the true state, to the principal. Of course, this report $\hat{\theta}$ is not necessarily identical to the truth θ. As in the classical contract theory literature [16], we assume that the principal cannot observe the agent's choice of action a (i.e., the hidden-action model).

To incentivize the agent to put in real effort in his exploration, the principal designs a contract p and offers it to the agent. We assume that the principal is trustworthy and will always fulfill the contract honestly. Both the principal and the agent are risk-neutral utility maximizers. Now, we introduce the definition of their utilities, which serve as the guidelines for contract design in this model. Define the mapping $v : \Theta \times \Theta \to \mathbb{R}^+$ such that $v(\hat{\theta}, \theta)$ is the benefit the principal gains from receiving the report $\hat{\theta}$ when the true state is θ. Note that v can be seen as an $m \times m$-dimensional non-negative matrix, so we also refer to it as a value matrix. In this section, we make the following assumption on the values of v:

Assumption 1. *The value matrix v satisfies*

$$v(\theta, \theta) - v(\theta', \theta) \geq 0, \quad \forall \theta, \theta' \in \Theta.$$

Under Assumption 1, the truthful report is the most beneficial to the principal among all possible reports. The principal attempts to elicit an accurate report via a carefully designed contract p. We consider the contract p to also be an $m \times m$-dimensional matrix, where $p(\hat{\theta}, \theta)$ is the payment delivered to the agent for his report $\hat{\theta}$ when the true state is θ. A payment mechanism contingent on both the report $\hat{\theta}$ and the truth θ is reasonable since in this section, we consider a scenario where both players can observe the true state θ after the agent submits his report $\hat{\theta}$. Following the literature on contract design [10], we focus on non-negative contracts, i.e., $p(\hat{\theta}, \theta) \geq 0, \forall \hat{\theta}, \theta \in \Theta$ (limited liability).

Given the offered contract p, the agent chooses his action and report to maximize his utility. The agent's strategy involves an effort investment strategy $s \in \Delta_{[n]}$ (we define Δ_Ω as the probability simplex on a finite set Ω) and a reporting strategy $r \in \mathcal{R}_0$, where $\mathcal{R}_0$ is the set of all possible reporting strategies and will be defined later. s_a is the probability of choosing action a. Let $\mathsf{suc} \in \{0, 1\}$ be an indicator variable of whether the exploration is successful. $\mathsf{suc} = 1$ if it is successful and $\mathsf{suc} = 0$ if not. The strategy r maps from a, suc, and the true state θ to a distribution over all possible reports. Noting that the report should be independent of θ when the exploration fails, we define

$$\mathcal{R}_0 = \Big\{ r : [n] \times \{0, 1\} \times \Theta \to \Delta_\Theta \;\Big|\; r(a, 0, \theta) = r(a, 0, \theta'), \\ \forall a \in [n], \theta, \theta' \in \Theta \Big\}.$$

Let $\mathrm{Bin}(q)$ denote the Bernoulli distribution with mean q. The agent's expected utility u_{0A} depending on his strategy (s, r) can be written as

$$u_{0A}(s, r) = \mathbb{E}_{\theta\sim\mathbb{P}_\theta, a\sim s, \mathsf{suc}\sim\mathrm{Bin}(q_a), \hat{\theta}\sim r(a,\mathsf{suc},\theta)} \left[p(\hat{\theta}, \theta) - c_a \right].$$

We will show in Proposition 1 that the following pure strategy is the agent's optimal strategy given p, breaking ties towards the benefit of the principal:

$$a = \arg\max_{a'\in[n]} - c_{a'} + q_{a'} \sum_{\theta\in\Theta} P_\theta p(\bar{r}(\theta), \theta) \\ + (1 - q_{a'}) \sum_{\theta\in\Theta} P_\theta p(\underline{r}, \theta), \tag{1}$$

$$\bar{r}(\theta) = \arg\max_{\theta'\in\Theta} p(\theta', \theta), \quad \forall \theta \in \Theta, \tag{2}$$

$$\underline{r} = \arg\max_{\theta'\in\Theta} \sum_{\theta\in\Theta} P_\theta p(\theta', \theta). \tag{3}$$

In this strategy, when the exploration is successful, he reports $\bar{r}(\theta)$ which maximizes the payment given the true state θ. When the exploration fails, he reports $\underline{r}$ which maximizes the expected payment given the prior distribution of θ. The agent's action a is the one that maximizes his expected utility.

Proposition 1. *The pure strategy $\left(a, \{\bar{r}(\theta)\}_{\theta\in\Theta}, \underline{r}\right)$ defined in (1), (2), and (3) achieves an expected agent utility of* $\max_{(s,r)\in\Delta_{[n]}\times\mathcal{R}_0} u_{0A}$ *(s, r), given any contract p.*

Now that the agent's behavior is determined by the contract p, we can write the principal's expected utility as

$$u_{0P}(p) = \mathbb{E}_{\theta\sim\mathbb{P}_\theta, \mathsf{suc}\sim\mathrm{Bin}(q_a)} \left[v(\hat{\theta}, \theta) - p(\hat{\theta}, \theta) \right] = \\ \sum_{\theta\in\Theta} P_\theta \Big[q_a \big(v(\bar{r}(\theta), \theta) - p(\bar{r}(\theta), \theta) \big) + (1 - q_a) \big(v(\underline{r}, \theta) - p(\underline{r}, \theta) \big) \Big],$$

noting that the agent's report $\hat{\theta} = \bar{r}(\theta)\mathbb{I}\{\mathsf{suc} = 1\} + \underline{r}\mathbb{I}\{\mathsf{suc} = 0\}$. In summary, the contract design problem faced by the principal can be formalized as a program:

$$\text{(P1)} \quad \max_p \quad u_{0P}(p)$$

$$s.t. \quad \sum_{\theta\in\Theta} P_\theta \left[p(\bar{r}(\theta), \theta) - p(\underline{r}, \theta) \right] (q_a - q_{a'}) \geq c_a - c_{a'}, \forall a' \in [n] \tag{4}$$

$$p(\bar{r}(\theta), \theta) \geq p(\theta', \theta), \quad \forall \theta, \theta' \in \Theta, \tag{5}$$

$$\sum_{\theta\in\Theta} P_\theta p(\underline{r}, \theta) \geq \sum_{\theta\in\Theta} P_\theta p(\theta', \theta), \quad \forall \theta' \in \Theta, \tag{6}$$

$$p(\theta, \theta') \geq 0, \quad \forall \theta, \theta' \in \Theta,$$

$$\underline{r} \in \Theta, a \in [n], \bar{r}(\theta) \in \Theta, \quad \forall \theta \in \Theta.$$

Constraints (4), (5), and (6) correspond to the agent's pure strategy (1), (2), and (3), respectively. This program has $m + 2$ discrete variables, namely $a, \underline{r}$, and $\{\bar{r}(\theta)\}_{\theta\in\Theta}$. Although P1 becomes a linear program when a realization of these variables is fixed, there are

Algorithm 1: Computing An Optimal Solution to P1

Initialize p to be an $m \times m$-dimensional zero matrix
`saved-z` $\leftarrow$ `NULL`, `saved-obj` $\leftarrow$ `-Inf`
for $a \in [n]$ *and* $\underline{r} \in \Theta$ **do**
  Solve the following linear program: $\vec{z} \leftarrow$

$$\begin{aligned} \min_{\vec{h}} \quad & q_a \sum_{\theta \in \Theta} P_\theta h_\theta + (1 - q_a) P_{\underline{r}} h_{\underline{r}}, \\ s.t. \quad & P_{\underline{r}} h_{\underline{r}} \geq P_\theta h_\theta, \forall \theta \neq \underline{r}, \\ & (q_a - q_{a'}) \sum_{\theta : \theta \neq \underline{r}} P_\theta h_\theta \geq c_a - c_{a'}, \forall a' \neq a, \\ & h_\theta \geq 0, \forall \theta \in \Theta. \end{aligned}$$

  `obj` $\leftarrow q_a \sum_{\theta \in \Theta} P_\theta \big(v(\theta, \theta) - z_\theta\big) + (1 - q_a) \big[\sum_{\theta \in \Theta} P_\theta v(\underline{r}, \theta) - P_{\underline{r}} z_{\underline{r}} \big]$
  if `obj > saved-obj` **then**
    `saved-obj`, `saved-z` $\leftarrow$ `obj`, $\vec{z}$
  end
end
$\vec{z} \leftarrow$ `saved-z`
for $\theta \in \Theta$ **do**
  $p(\theta, \theta) \leftarrow z_\theta$
end
Output: p the solution to P1

$n \cdot m^{m+1}$ possible realizations. Thus, it is prohibitive to solve this program by enumerating all realizations of the discrete variables.

3.2 Computing the Optimal Contract Efficiently

We introduce an efficient solution for P1 and present it in Algorithm 1. It solves $n \cdot m$ linear programs. Each program has m variables and $2m + n - 2$ inequality constraints. We present the following result justifying the correctness of our method.

Theorem 2. *Algorithm 1 computes an optimal solution to P1.*

The key ingredient of its proof is demonstrating that there exists an optimal solution to P1 within a small but reasonable family of non-negative $m \times m$-dimensional matrices.

Definition 3 (Diagonal Contract). We say a contract p is diagonal if and only if it satisfies

$$p(\theta', \theta) = 0, \quad \forall \theta, \theta' \in \Theta \quad s.t. \quad \theta' \neq \theta.$$

For any diagonal contract, the entries are zero except for the ones exactly on the diagonal, so the agent only gets paid when his prediction is correct. In the following lemma, we show that to solve the program P1, it suffices to consider only the diagonal contracts.

Lemma 1. *Let $p_0 : \Theta \times \Theta \to \mathbb{R}^+$ be any non-negative contract. Then there exists a non-negative diagonal contract whose objective value is no smaller than that of p_0.*

Proof Sketch of Lemma 1. We show a transformation from p_0 into a non-negative diagonal contract p_3 that yields non-decreasing principal utility, involving three steps:

(i) $p_0 \longrightarrow p_1$. For any $\hat{\theta}, \theta \in \Theta$, if $\hat{\theta}$ is neither $\underline{r}$ nor $\bar{r}(\theta)$, we set $p_1(\hat{\theta}, \theta) = 0$. Otherwise, we set $p_1(\hat{\theta}, \theta) = p_0(\hat{\theta}, \theta)$. This modification does not change the agent's choice of effort or his reporting strategy. We can also show that the principal's utility remains unchanged.

(ii) $p_1 \longrightarrow p_2$. We shift the highest payment on each column of p_1 to its diagonal and reduce all of p_1's non-diagonal elements in the row indexed by $\underline{r}$ to 0. This gives a diagonal contract p_2. Let $\theta^* = \arg\max_{\theta' \in \Theta} \sum_{\theta \in \Theta} P_\theta p_2(\theta', \theta) = \arg\max_{\theta \in \Theta} P_\theta p_2(\theta, \theta)$ denote agent's actual report at exploration failure in terms of p_2. Then, let $\theta_0 \in \arg\max_{\theta' \in \Theta} \sum_{\theta \in \Theta} P_\theta v(\theta', \theta)$ denote the principal's most desired report when agent's exploration fails.

(iii) $p_2 \longrightarrow p_3$. We need to adjust p_2 to match θ^* with θ_0. We 'swap' $p_2(\theta^*, \theta^*)$ and $p_2(\theta_0, \theta_0)$ to induce p_3, such that

$$p_3(\theta_0, \theta_0) = \frac{P_{\theta^*}}{P_{\theta_0}} p_2(\theta^*, \theta^*),$$
$$p_3(\theta^*, \theta^*) = \frac{P_{\theta_0}}{P_{\theta^*}} p_2(\theta_0, \theta_0),$$

to incentivize the agent to report θ_0 when his exploration fails. It can be shown that our modification will not reduce the principal's utility.

Figure 2 illustrates an example of this transformation process. □

The proof of Theorem 2 is straightforward with Lemma 1, thus we defer it to the technical appendix. An important implication of Lemma 1 is that, in the model where the true state is finally accessible to the principal, the objectives of optimizing the principal's utility and encouraging the agent to tell the truth are fully compatible. When the principal can incentivize the agent to both work hard and provide honest information, her utility will be maximized.

4 CONTRACTING ON NOISY FEEDBACK

In the previous section, we assumed that the principal can ultimately observe the true state of the environment, θ, and make payments to the agent accordingly. However, accessing this information is often infeasible in practice. For instance, consider a client (the Principal) seeking advice from an expert (the Agent) about the financial market. While she can easily gather public data like stock prices, it is impractical for her to immediately verify the expert's recommendations due to either her limited financial expertise or the extensive time and effort required for validation. In this section, we explore the possibility of incentivizing the agent to investigate the true state and truthfully present his findings, even when the agent's report cannot be directly validated.

4.1 The Model

The Principal-Agent model used here is similar to that in the last section. There are m possible environment states, i.e., $m = |\Theta|$. The Nature samples θ from a prior distribution $\mathbb{P}_\theta$. A principal delegates the truth exploration task to an agent. They both know the prior distribution $\mathbb{P}_\theta$. The agent first chooses his level of effort and then presents his report $\hat{\theta} \in \Theta$ to the principal.

Unlike our previous model, we consider a scenario where the principal faces a noisy feedback $X \in \mathbb{R}$ that is jointly determined by

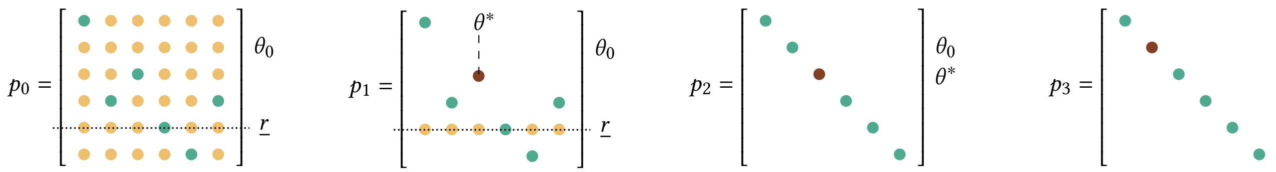

Figure 2: An example that demonstrates how to transform p_0 into a diagonal contract p_3 in the proof of Lemma 1. The elements in p_0 at positions $(\bar{r}(\theta), \theta)$ for all $\theta \in \Theta$ are shown as green nodes, while the others are shown as yellow nodes. The element $p_1(\bar{r}(\theta^*), \theta^*)$ is shown as a brown node. Each empty slot in the matrices indicates that the corresponding payment is 0.

the agent's report $\hat{\theta}$ and the true state θ, but can never directly observe θ. For example, X could be the market value of the principal's portfolio (adjusted after considering the agent's market research report) on a particular trading day, while θ can be the true state of the market. We assume that the random feedback admits an expectation captured by the value matrix v. Specifically,

$$X = v(\hat{\theta}, \theta) + E, \quad E \sim F_\eta,$$

where E is a zero-mean stochastic noise with a cumulative distribution function F_η. Due to the presence of random noise, it is difficult for the principal to infer the true underlying state solely from observing X. Here, we make a slightly stronger assumption (Assumption 2) about the value matrix compared to Section 3. This assumption implies that the principal's profit from receiving truthful reports is at least δ higher than that from receiving non-truthful reports, where δ is a known constant.

Assumption 2. *For a constant $\delta \geq 0$, the value matrix v satisfies*

$$v(\theta, \theta) - v(\theta', \theta) \geq \delta, \quad \forall \theta, \theta' \in \Theta.$$

As in the previous model, the principal is interested in incentivizing the agent to exert effort in his truth investigation via a contract p. Since the only information the principal has is the agent report $\hat{\theta}$ and the random feedback X, it is reasonable to focus on the set of contracts $\mathcal{P} := \{p : \Theta \times \mathbb{R} \to \mathbb{R}^+\}$. In this section, we consider a simpler setting where the agent has only two levels of effort: $c_1 = 0, q_1 = 0$ and $c_2 = c, q_2 = 1$ for a constant $c > 0$. That is, the agent can either uncover the truth θ at a price c or choose to receive no information without any charge.

Given the offered contract p, the agent chooses his action and report to maximize his utility. $s \in \Delta_{[2]}$ and $r \in \mathcal{R}$ are the agent's effort investment strategy for truth exploration and reporting strategy, respectively. In this section, we define the set of all possible agent reporting strategies

$$\mathcal{R} = \left\{ r : [2] \times \Theta \to \Delta_\Theta \,\middle|\, r(1, \theta) = r(1, \theta'), \forall \theta, \theta' \in \Theta \right\}.$$

The agent's expected utility u_A can be written as

$$u_A(s, r) = \mathbb{E}_{\theta \sim \mathbb{P}_\theta, a \sim s, \hat{\theta} \sim r(a, \theta), E \sim F_\eta} \left[p(\hat{\theta}, X) - c_a \right].$$

Let $F_{X|\hat{\theta},\theta}$ be the distribution function of X when the report is $\hat{\theta}$ and the truth is θ. It can be shown that the following pure strategy is the agent's optimal strategy given p, breaking ties towards the benefit of the principal:

$$a = \begin{cases} 2, & \text{if } \sum_{\theta \in \Theta} P_\theta \int_{\mathbb{R}} p(\bar{r}(\theta), x) dF_{X|\bar{r}(\theta),\theta} - c \geq \\ & \quad \sum_{\theta \in \Theta} P_\theta \int_{\mathbb{R}} p(\underline{r}, x) dF_{X|\underline{r},\theta}, \\ 1, & \text{otherwise}, \end{cases} \tag{7}$$

$$\bar{r}(\theta) = \arg\max_{\theta' \in \Theta} \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta}, \quad \forall \theta \in \Theta, \tag{8}$$

$$\underline{r} = \arg\max_{\theta' \in \Theta} \sum_{\theta \in \Theta} P_\theta \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta}, \tag{9}$$

where the integrals are w.r.t. the variable x. In this strategy, the agent reports $\bar{r}(\theta)$ when he invests effort in exploration and reports $\underline{r}$ when he does not exert any effort. The following result justifies the optimality of this strategy.

Proposition 2. *The pure strategy $(a, \{\bar{r}(\theta)\}_{\theta \in \Theta}, \underline{r})$ defined in (7), (8) and (9) achieves an expected agent utility of $\max_{(s,r) \in \Delta_{[2]} \times \mathcal{R}} u_A(s, r)$, given any contract $p \in \mathcal{P}$.*

Now that the agent's behavior is determined by the contract p, we can write the principal's expected utility as

$$\begin{aligned} u_P(p) &= \mathbb{E}_{\theta \sim \mathbb{P}_\theta, E \sim F_\eta} \left[v(\hat{\theta}, \theta) - p(\hat{\theta}, X) \right] \\ &= \sum_{\theta \in \Theta} P_\theta \left[v(\hat{\theta}, \theta) - \int_{\mathbb{R}} p(\hat{\theta}, x) dF_{X|\hat{\theta},\theta} \right], \end{aligned}$$

where the agent's report is given by $\hat{\theta} = \bar{r}(\theta)\mathbb{I}\{a = 2\} + \underline{r}\mathbb{I}\{a = 1\}$. In summary, the contract design problem faced by the principal can be formalized as a program:

$$\begin{aligned} \text{(P2)} \quad & \max_{p \in \mathcal{P}} \quad u_P(p) \\ s.t. \; & \sum_{\theta \in \Theta} P_\theta \left[\int_{\mathbb{R}} p(\bar{r}(\theta), x) dF_{X|\bar{r}(\theta),\theta} - \int_{\mathbb{R}} p(\underline{r}, x) dF_{X|\underline{r},\theta} \right] \\ & \quad \times (q_a - q_{a'}) \geq c_a - c_{a'}, \quad \forall a' \in [2] \\ & \int_{\mathbb{R}} p(\bar{r}(\theta), x) dF_{X|\bar{r}(\theta),\theta} \geq \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta}, \quad \forall \theta, \theta' \in \Theta, \\ & \sum_{\theta \in \Theta} P_\theta \left[\int_{\mathbb{R}} p(\underline{r}, x) dF_{X|\underline{r},\theta} - \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta} \right] \geq 0, \forall \theta' \in \Theta, \\ & \underline{r} \in \Theta, a \in [2], \bar{r}(\theta) \in \Theta, \quad \forall \theta \in \Theta. \end{aligned}$$

To simplify this program, we make the following observation. For any contract $p \in \mathcal{P}$, if it incentivizes zero agent effort, i.e., $a = 1$, the principal's utility is at most

$$u_P(p) \leq \sum_{\theta \in \Theta} P_\theta v(\underline{r}, \theta) \leq \max_{\hat{\theta} \in \Theta} \sum_{\theta \in \Theta} P_\theta v(\hat{\theta}, \theta) =: \underline{u}_P.$$

This utility upper bound can be achieved by a zero-payment contract p $s.t.$ $p(\hat{\theta}, x) = 0, \forall \hat{\theta} \in \Theta, x \in \mathbb{R}$. One interpretation of the zero-payment contract is that when the utility derived from delegating the truth exploration task is insufficient, it is more advantageous for the principal to make a prediction herself based on her prior knowledge $\mathbb{P}_\theta$. This observation implies that solving P2 only requires finding an optimal positive-effort-incentivizing contract and comparing its utility against the baseline principal utility $\underline{u}_P$.

4.2 Our Contract Design

In this part, we propose a feasible solution to P2. We aim to design a contract that encourages the agent to truthfully report his findings while maximizing the principal's net utility. Before presenting our design in detail, we first introduce a lower bound on the expected payment, as it provides insight into our contract design. This lower bound characterizes the maximum principal utility that any contract can generate under the truthful reporting constraint.

Proposition 3. *Suppose the noise distribution F_η has a probability density function ϕ_η. A lower bound for the payment that the principal has to make to incentivize positive effort and truth-telling is given by the following linear program*

$$(L1) \quad LB := \min_{\vec{t}} \sum_{\theta \in \Theta} t_\theta$$

$$s.t. \quad \sum_{\theta \in \Theta} t_\theta - c \geq \Big[\inf_{s \in \mathbb{R}} \sum_{\theta \in \Theta} \alpha_{\theta,\theta'}(s) \Big] t_{\theta'}, \quad \forall \theta' \in \Theta, \tag{10}$$

$$t_\theta \geq \Big[\inf_{s \in \mathbb{R}} \alpha_{\theta,\theta'}(s) \Big] t_{\theta'}, \quad \forall \theta, \theta' \in \Theta, \tag{11}$$

$$t_\theta \geq 0, \quad \forall \theta \in \Theta, \tag{12}$$

where $\alpha_{\theta,\theta'}(s) := \big[P_\theta \phi_\eta\big(s - v(\theta', \theta)\big)\big] / \big[P_{\theta'} \phi_\eta\big(s - v(\theta', \theta')\big)\big]$.

We explain this result. For any contract p, define $\vec{y}$ $s.t.$ $y_\theta = P_\theta \int_{\mathbb{R}} p(\theta, x) dF_{X|\theta,\theta}, \forall \theta \in \Theta$. (10) and (11) can be shown to be relaxed versions of the positive effort and truth-telling constraints, respectively. Thus, if p incentivizes positive effort and truth-telling, $\vec{y}$ must satisfy (10) and (11). This leads to $\sum_{\theta \in \Theta} y_\theta \geq \text{LB}$, as $\vec{y}$ is a feasible solution to this program. We complete the proof by observing that $\sum_{\theta \in \Theta} y_\theta$ is precisely the expected payment of p. Intuitively, for p to generate principal utility close to the optimum, the original positive effort and truth-telling constraints it induces should be close to the relaxed versions (10) and (11).

We propose contracts of the form:

$$p(\hat{\theta}, x) = \begin{cases} B_{\hat{\theta}}, & \text{if } |x - v(\hat{\theta}, \hat{\theta})| \leq \rho_{\hat{\theta}}, \\ 0, & \text{otherwise,} \end{cases} \quad \forall x \in \mathbb{R}, \forall \hat{\theta} \in \Theta.$$

B_θ, ρ_θ are non-negative parameters for any $\theta \in \Theta$. Since $p(\hat{\theta}, \cdot)$ approaches a Dirac pulse at $x = v(\hat{\theta}, \hat{\theta})$ when $\rho_{\hat{\theta}} \to 0$, we refer to this type of contract as **Bounded Dirac Delta** (BDD) contracts. The detailed computation process is given in Algorithm 2. This algorithm first computes the weighted expected payment associated with each true state, represented as the vector $\vec{z}$. It then sets the values of B_θ to align the expected payments of p with $\vec{z}$. Our proposed contract has three advantages:

(1) **Ex-Post Boundedness.** There exists an ex-post budget constraint B such that $B_\theta \leq B, \forall \theta \in \Theta$, which is necessary for a contract to be feasible in practice.

Algorithm 2: Bounded Dirac Delta (BDD) Contract

Input: $\{\rho_\theta\}_{\theta \in \Theta}$ payment radius

Initialize $p(\theta, x) = 0, \forall \theta \in \Theta, x \in \mathbb{R}$

Solve the following linear program: $\vec{z} \leftarrow$

$$\min_{\vec{t}} \sum_{\theta \in \Theta} t_\theta$$

$$s.t. \quad \sum_{\theta \in \Theta} t_\theta - c \geq \left[\sum_{\theta \in \Theta} \frac{P_\theta \int_{v(\theta',\theta') - v(\theta',\theta) - \rho_{\theta'}}^{v(\theta',\theta') - v(\theta',\theta) + \rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \right] \cdot t_{\theta'}, \forall \theta' \in \Theta,$$

$$t_\theta \geq \left[\frac{P_\theta \int_{v(\theta',\theta') - v(\theta',\theta) - \rho_{\theta'}}^{v(\theta',\theta') - v(\theta',\theta) + \rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \right] t_{\theta'}, \forall \theta, \theta' \in \Theta,$$

$$t_\theta \geq 0, \forall \theta \in \Theta.$$

if $\sum_{\theta \in \Theta} z_\theta \leq \sum_{\theta \in \Theta} P_\theta v(\theta, \theta) - \underline{u}_P$ **then**

  for $\theta \in \Theta$ **do**

    Compute $B_\theta \leftarrow z_\theta / \big[P_\theta \int_{-\rho_\theta}^{\rho_\theta} dF_\eta\big]$

    Update the contract

$$p(\theta, x) \leftarrow \begin{cases} B_\theta, & \text{if } |x - v(\theta, \theta)| \leq \rho_\theta, \\ 0, & \text{otherwise,} \end{cases} \quad \forall x \in \mathbb{R}$$

  end

end

Output: p the contract

(2) **Near Optimality.** It can be shown that, under mild assumptions on F_η and when B is large, the positive effort and truth-telling constraints approach (10) and (11).

(3) **Computational Simplicity.** We note that for BDD contracts, the positive effort and truth-telling constraints are linear constraints without relaxation. Thus, the optimal payment can be efficiently computed by solving a linear program in Algorithm 2, which is similar to L1 in Proposition 3.

Define $\bar{l} = \max_{\theta,\theta' \in \Theta} |v(\theta, \theta) - v(\theta, \theta')|$, $\underline{l} = \min_{\theta \neq \theta'} |v(\theta, \theta) - v(\theta, \theta')|$. To facilitate our theoretical analysis of this method, we make the following assumptions on the noise distribution F_η.

Assumption 3. *The noise distribution F_η has a probability density function ϕ_η satisfying:*

- *ϕ_η is symmetric: $\phi_\eta(x) = \phi_\eta(-x), \forall x \in \mathbb{R}$,*
- *ϕ_η is monotonically non-increasing in $\mathbb{R}^+$: $\phi_\eta(x_1) \geq \phi_\eta(x_2)$, $\forall 0 \leq x_1 \leq x_2$,*
- *ϕ_η is L-Lipschitz on $[-\bar{l}, \bar{l}]$,*
- *$\phi_\eta(x - d)/\phi_\eta(x) \geq \phi_\eta(-d)/\phi_\eta(0), \forall x \leq 0, d \geq 0$.*

It can be validated that the family of Laplace distributions satisfies this assumption.

Definition 4 (Laplace Distribution). For any constant $\lambda > 0$, a random variable has a zero-mean Laplace distribution, denoted $\text{Laplace}(0, 1/\lambda)$, if its probability density function is

$$\phi(x) = \frac{\lambda}{2} \exp\big(-\lambda|x|\big), \quad \forall x \in \mathbb{R}.$$

We formally present our theoretical result for the proposed contract in Theorem 5. Say p is a BDD contract computed by Algorithm

2, we show that the gap between the highest possible utility generated by a truth-telling incentivizing contract and the utility yielded by p converges to 0 at a speed of $O(1/B)$ when B increases. That is, our contract is a good approximation of the optimal truth-telling incentivizing contract. In the remainder of this subsection, we outline the proof idea for this theorem.

THEOREM 5. *Let p be a contract generated by Algorithm 2 with inputs*

$$\rho_\theta = F_\eta^{-1}\left[\frac{1}{2} + \frac{cP_\theta^{-1}B^{-1}}{1 - \max_{\theta'\in\Theta}\sum_{\tilde{\theta}\in\Theta} P_{\tilde{\theta}} \frac{\phi_\eta(v(\theta',\theta') - v(\theta',\tilde{\theta}))}{\phi_\eta(0)}}\right]$$

for any $\theta \in \Theta$, where $F_\eta^{-1}(y) = \min\{x \in \mathbb{R} \mid F_\eta(x) = y\}, \forall y \in (0,1)$. Suppose Assumption 2, 3 hold. Then $\exists B_0 \in \mathbb{R}^+, \forall B \geq B_0$, contract p has the following properties:

1. *The budget constraint is never violated: $B_\theta \leq B, \forall \theta \in \Theta$.*
2. *The agent is incentivized to report the true state that he observes after the exploration.*
3. *The difference between the principal utility generated by any truth-telling incentivizing contract p_0 and the utility induced by contract p is upper bounded as*

$$u_P(p_0) - u_P(p) \leq \frac{12\phi_\eta(0)^2 Lc^2}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^4 \underline{P}^2 B}.$$

We introduce the proof idea of Theorem 5. The first two claims in Theorem 5 can be easily validated. For the third one, we notice that since both p_0 and p incentivize the agent to report honestly, they generate the same expected profit $\sum_{\theta\in\Theta} P_\theta v(\theta,\theta)$ for the principal. Thus, it suffices to consider the difference between their expected payments. The key to upper-bounding this difference is as follows:

Since the payment of p is defined by the optimal value of another linear program (in Algorithm 2), which can be shown to be similar to L1, we derive a sensitivity analysis result to demonstrate that the optimal values of both programs are close to each other. This completes the proof of Theorem 5.

LEMMA 2. *Consider the following two linear programs,*

$$\begin{aligned}
(O1) \quad OPT := \min_{\vec{t}} \quad & \sum_{\theta\in\Theta} t_\theta \\
s.t. \quad & b_{\theta'} t_{\theta'} - \sum_{\theta\in\Theta} t_\theta + c \leq 0, \quad \forall \theta' \in \Theta, \\
& a_{\theta,\theta'} t_{\theta'} - t_\theta \leq 0, \quad \forall \theta, \theta' \in \Theta, \theta \neq \theta', \qquad (13) \\
& - t_\theta \leq 0, \quad \forall \theta \in \Theta
\end{aligned}$$

and O2 (OPT' is its optimal value). O2 has the same structure as O1, except that in O2, $a'_{\theta,\theta'}$ replaces $a_{\theta,\theta'}$ and $b'_{\theta'}$ replaces $b_{\theta'}$ for any $\theta, \theta' \in \Theta$. We require $0 \leq a_{\theta,\theta'} \leq a'_{\theta,\theta'} < P_\theta/P_{\theta'}, \forall \theta, \theta' \in \Theta, \theta \neq \theta'$ and $0 \leq b_\theta \leq b'_\theta < 1/P_\theta, \forall \theta \in \Theta$. If there exist series of small positive constants $\{\epsilon_{\theta,\theta'}\}_{\theta,\theta'\in\Theta}, \{\kappa_\theta\}_{\theta\in\Theta}$ such that $0 \leq a'_{\theta,\theta'} - a_{\theta,\theta'} \leq \epsilon_{\theta,\theta'}, \forall \theta, \theta' \in \Theta, \theta \neq \theta'$ and $0 \leq b'_\theta - b_\theta \leq \kappa_\theta, \forall \theta \in \Theta$, then we have that

$$\frac{OPT'}{OPT} \leq \left[1 - \frac{(1-\lambda) + \max_{\theta\in\Theta}\kappa_\theta}{1 - \max_{\theta\in\Theta} b_\theta P_\theta}\right]^{-1},$$

where $\lambda \in (0,1)$ is a constant such that

$$1 - \lambda \leq \max_{\theta\neq\theta'} \frac{\epsilon_{\theta,\theta'}}{[P_\theta - a'_{\theta,\theta'} P_{\theta'}] + \epsilon_{\theta,\theta'}}.$$

PROOF SKETCH OF LEMMA 2. Assume that $\vec{t}^*$ is an optimal solution to O1. Our goal is to construct a solution to O2 whose sum is as close to $\sum_{\theta\in\Theta} t^*_\theta$ as possible. The first step is to construct an intermediate solution $\vec{y}$ from $\vec{t}^*$ such that $\vec{y}$ satisfies the corresponding constraint of (13) in O2. We achieve this by finding a vector $\vec{t}'$ that satisfies this constraint and defining $\vec{y} = \lambda\vec{t}^* + (1-\lambda)\vec{t}'$ for a constant $\lambda \in (0,1)$. We show that a $\lambda \to 1$ can guarantee a $\vec{y}$ we want. The second step is to find a constant $\mu > 1$ such that $\mu\vec{y}$ is a feasible solution to O2. We conclude this proof by showing that there exists a $\mu \to 1$ which satisfies our requirement. □

Remark. One may notice that the above analysis does not apply to the Gaussian distribution due to its violation of Assumption 3. In the appendix, we present an analysis for another family of noise distributions, including Gaussian distributions. We prove an upper bound for $u_P(p_0) - u_P(p)$ that converges to 0 as the Gaussian variance σ^2 decreases, which is verified by our experiments.

4.3 Dropping the Honest Reporting Constraint

In Section 4.2, our proposed BDD contract was evaluated against the best possible contract that must encourage the agent to report honestly, rather than the optimal contract that purely maximizes the principal's net utility. In this context, one might naturally ask the following question:

Does the optimal contract necessarily incentivize truth-telling, so that the two benchmarks above are simply equivalent?

We demonstrate that the answer is negative by presenting the following counterexample. The principal's value matrix is set to be

$$v = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$$

with the dimension $m = 4$. Let θ_i denote the true state corresponding to the i-column of the value matrix. We set the prior distribution $P_\theta = \frac{1}{m}, \forall \theta \in \Theta$. The agent's cost of positive effort $c = 1$. The noise distribution F_η is a Laplace distribution with a sufficiently large λ.

The solution to the linear program L1 for this problem instance is $\vec{y} = \left[\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3}\right]$, indicating that the expected payment must be at least $\frac{5}{3}$ to incentivize both positive effort and truth-telling. This solution implies that the expected payment when the true state is θ_1 and the agent tells the truth is equal to that when the true state is θ_2 and the agent tells the truth. Such equality is necessary for the adopted contract to incentivize truth-telling, given the structure of the value matrix v. If the payment associated with θ_1 were lower than that with θ_2, the agent would gain more by reporting $\hat{\theta} = \theta_2$ when the true state is θ_1, thereby violating the truth-telling constraint. However, we can construct a contract that breaks this constraint while delivering less payment, thus showing that the optimal contract does not incentivize truth-telling. The constructed contract p implements $\vec{y}' = \left[\frac{1}{6} - \frac{13}{500}, \frac{1}{6} + \frac{3}{50}, \frac{2}{3} - \frac{1}{10}, \frac{2}{3} - \frac{1}{10}\right]$, a slightly perturbed version of $\vec{y}$. This contract does not incentivize truth-telling (the agent reports $\hat{\theta} = \theta_2$ when the true state is θ_1) but reduces the expected payment to $\frac{5}{3} - \frac{2}{25} < \frac{5}{3}$.

Although the answer is negative in general, we do identify a sufficient condition for the two benchmarks to be equivalent: Assumption 2 holds with a sufficiently large δ. Recall that given this assumption, for any realized true state θ, the principal profit decreases by at least δ if the agent is incentivized to report any $\hat{\theta} \neq \theta$. When the reduced cost can never offset the decreased profit, the optimal contract must incentivize both positive effort and truth-telling simultaneously. We formalize this intuition in Theorem 6 under the following assumption. Define $\bar{v} = \max_{\theta \in \Theta} v(\theta, \theta)$ and $\underline{v} = \min_{\theta \in \Theta} v(\theta, \theta)$.

Assumption 4. *The noise distribution F_η has a probability density function ϕ_η satisfying:*

- *ϕ_η is symmetric: $\phi_\eta(x) = \phi_\eta(-x)$, $\forall x \in \mathbb{R}$,*
- *ϕ_η is monotonically non-increasing in $\mathbb{R}^+$: $\phi_\eta(x_1) \geq \phi_\eta(x_2)$, $\forall 0 \leq x_1 \leq x_2$.*

Theorem 6. *For any constants $\lambda, \nu \in (0, 1)$ s.t. $\lambda > \nu$, suppose Assumption 2 with a sufficiently large δ such that the following conditions holds: (a) $\delta \geq \lambda \bar{v} \bigvee \frac{1}{2\underline{P}}\left[c + \sqrt{c^2 + 16c\bar{v}\underline{P}}\right]$, (b) $\phi_\eta(\delta - \nu\bar{v}) \leq \frac{8}{8+\lambda}\frac{\underline{v}}{\bar{v}}\phi_\eta(0)$, (c) $\phi_\eta(\delta/2 - \nu\bar{v}) \leq \frac{\lambda}{8+\lambda}\underline{P}\phi_\eta(0)$ and Assumption 4 hold. Given any contract p_0 such that $\exists \theta \in \Theta, \bar{r}(\theta) \neq \theta$, there exists a contract p satisfying that*

(1) p is ex-post upper bounded in its value,
(2) p incentivizes honest reporting,
(3) $u_P(p) \geq u_P(p_0)$.

5 EMPIRICAL RESULTS

In this section, we present our experiment results. Our machine is a PC running Windows 11, equipped with an AMD Ryzen 9 5900X 12-Core Processor and an NVIDIA GeForce RTX 3060 GPU.

5.1 Evaluating the BDD Contract

In this experiment, we evaluate the principal utility generated by the proposed BDD contract. We set $m = 5$, $c = 0.2$ and randomly generate a 5×5-dimensional value matrix v. We set a uniform prior distribution over Θ. We consider various settings of the noise distribution F_η:

- Laplace distribution with $\lambda = 10, 20, 30$.
- Gaussian distribution with $\sigma = 0.04, 0.03, 0.02$.

The results are shown in Figure 3. In each subfigure, the horizontal axis is the budget constraint B. In addition to the utility generated by the positive-effort incentivizing BDD contract, we also illustrate the utility upper bound $\sum_{\theta \in \Theta} P_\theta v(\theta, \theta) - \text{LB}$ and the zero-effort utility $\underline{u}_P$ for comparison. Note that although $u_P(p)$ can be lower than $\underline{u}_P$ in subfigure (a), the utility generated by our Algorithm 2 is always at least $\underline{u}_P$. Our experiment demonstrates that the proposed BDD contract is a good approximation of the optimal truth-telling incentivizing contract, especially when the budget constraint B is large.

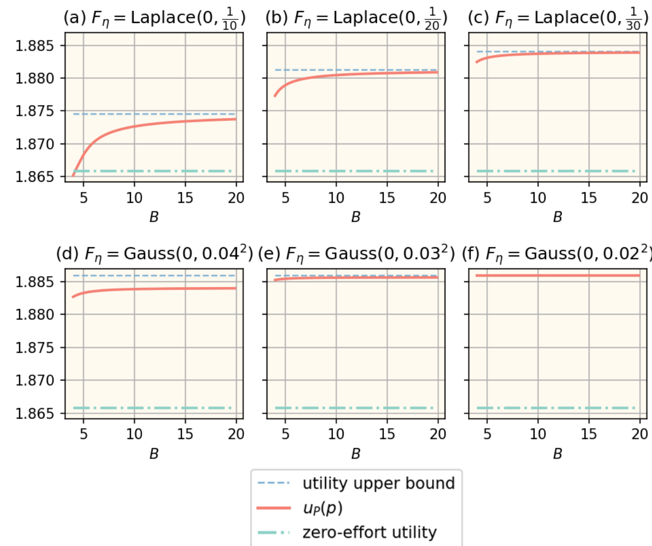

Figure 3: Expected principal utility generated by the proposed BDD contract. We consider two families of noise distributions. In (a)-(c) we set F_η to be the Laplace distribution, while in (d)-(f) we set it to be the Gaussian distribution. As the budget constraint B grows sufficiently large, the generated principal utilities approach the upper bounds. This demonstrates the near-optimality of our proposed contract.

6 CONCLUSION

In this paper, we propose a principal-agent problem in which a principal incentivizes an agent to undertake a costly exploration of the truth and report the findings through a payment contract. For different setups of feedback information that the principal can use to assess the quality of the agent's report, we demonstrate the importance and efficiency of encouraging the agent to submit honest reports and design our contract solutions accordingly. All omitted proofs in this paper can be found in the technical appendix.

Future Work. We believe there is still much to explore within the delegated truth exploration framework: (1) In this paper, we assume a principal with extensive knowledge, such as the prior distribution of the environment state, the agent's action set (including success rates and costs), and so on. Inspired by a recent work [6] on proper scoring rules, we find it interesting to extend our discussion to a partial knowledge setting. Moreover, several recent works have focused on the repeated principal-agent interaction setting, e.g., [2, 4, 7, 17]. It would also be valuable to extend our framework to a multi-round variant where the game starts with unknown parameters. (2) In this paper, we assume strict incentive compatibility, meaning the agent is always a utility maximizer. However, the agent may also be willing to follow the principal's suggestion, provided the utility is sufficiently close to optimal. We term this the approximate incentive compatibility assumption. Contract design problems in this context remain unexplored.

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A MISSING PROOFS FOR SECTION 3

Proof of Proposition 1. For any $(s,r) \in \Delta_{[n]} \times \mathcal{R}_0$, $u_{0A}(s,r)$ can be expanded as

$$\begin{aligned}
u_{0A}(s,r) &= \sum_{\theta\in\Theta} P_\theta \sum_{a\in[n]} s_a \left[q_a \Big(\sum_{\hat\theta\in\Theta} p(\hat\theta,\theta) r_{\hat\theta}(a,1,\theta) - c_a \Big) + (1-q_a)\Big(\sum_{\hat\theta\in\Theta} p(\hat\theta,\theta) r_{\hat\theta}(a,0,\theta) - c_a \Big) \right] \\
&\le \sum_{\theta\in\Theta} P_\theta \sum_{a\in[n]} s_a \left[q_a \max_{\hat\theta\in\Theta} p(\hat\theta,\theta) + (1-q_a)\Big(\sum_{\hat\theta\in\Theta} p(\hat\theta,\theta) r_{\hat\theta}(a,0,\theta) \Big) - c_a \right] \\
&= \sum_{a\in[n]} s_a \left[q_a \sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta} p(\hat\theta,\theta) + (1-q_a) \sum_{\hat\theta\in\Theta} r_{\hat\theta}(a,0) \sum_{\theta\in\Theta} P_\theta p(\hat\theta,\theta) - c_a \right] \\
&\le \sum_{a\in[n]} s_a \left[q_a \sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta} p(\hat\theta,\theta) + (1-q_a) \max_{\hat\theta\in\Theta} \sum_{\theta\in\Theta} P_\theta p(\hat\theta,\theta) - c_a \right] \\
&\le \max_{a\in[n]} \left[q_a \sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta} p(\hat\theta,\theta) + (1-q_a) \max_{\hat\theta\in\Theta} \sum_{\theta\in\Theta} P_\theta p(\hat\theta,\theta) - c_a \right],
\end{aligned}$$

where we wrote $r(a,0) = r(a,0,\theta), \forall\theta\in\Theta$ by the definition of $\mathcal{R}_0$. Let (s^0, r^0) represent the agent strategy defined in (1), (2), and (3). It can be validated that

$$\sum_{\hat\theta\in\Theta} p(\hat\theta,\theta) r^0_{\hat\theta}(a,1,\theta) = p\big(\bar r(\theta),\theta\big) = \max_{\hat\theta\in\Theta} p(\hat\theta,\theta), \forall a\in[n], \theta\in\Theta,$$

$$\begin{aligned}
&\sum_{\theta\in\Theta} P_\theta \sum_{\hat\theta\in\Theta} p(\hat\theta,\theta) r^0_{\hat\theta}(a,0,\theta) = \sum_{\hat\theta\in\Theta}\sum_{\theta\in\Theta} P_\theta p(\hat\theta,\theta) r^0_{\hat\theta}(a,0,\theta) \\
&= \sum_{\theta\in\Theta} P_\theta p(\underline{r},\theta) = \max_{\hat\theta\in\Theta} \sum_{\theta\in\Theta} P_\theta p(\hat\theta,\theta), \forall a\in[n],
\end{aligned}$$

and furthermore,

$$\begin{aligned}
u_{0A}(s^0,r^0) &= \sum_{a\in[n]} s^0_a \left[q_a \sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta} p(\hat\theta,\theta) + (1-q_a) \max_{\hat\theta\in\Theta} \sum_{\theta\in\Theta} P_\theta p(\hat\theta,\theta) - c_a \right] \\
&= \max_{a\in[n]} \left[q_a \sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta} p(\hat\theta,\theta) + (1-q_a) \max_{\hat\theta\in\Theta} \sum_{\theta\in\Theta} P_\theta p(\hat\theta,\theta) - c_a \right] \\
&\ge \max_{(s,r)\in\Delta_{[n]}\times\mathcal{R}_0} u_{0A}(s,r)
\end{aligned}$$

by our upper bound derived at the beginning of this proof. □

Proof of Lemma 1. Without loss of generality, we assume that $c_a < c_{a'}$ for any $a, a' \in [n]$ such that $a < a'$. Consider an arbitrary contract p_0 such that $p_0(\theta',\theta) \ge 0$ for any $\theta',\theta\in\Theta$. We will show how to transform p_0 into a diagonal contract without reducing the principal's expected net utility. Figure 2 in the main paper summarizes the transformation procedures used in this proof. Recall that given p_0, the agent reports $\bar r(\theta) \in \arg\max_{\theta'\in\Theta} p_0(\theta',\theta)$ given the truth θ, the prior optimal report is denoted as $\underline{r} \in \arg\max_{\theta'\in\Theta} \sum_{\tilde\theta\in\Theta} P_{\tilde\theta} p_0(\theta',\tilde\theta)$, and the agent's action a maximizes his expected net utility $\sum_{\theta\in\Theta} P_\theta \Big[q_a p_0(\bar r(\theta),\theta) + (1-q_a) p_0(\underline{r},\theta) \Big] - c_a$, assuming that ties are broken towards the benefit of the principal. The tuple $\big(a, \{\bar r(\theta)\}_{\theta\in\Theta}, \underline{r}\big)$ summarizes the agent's strategy under the contract p.

The first modification on p_0 is to set $p_0(\theta',\theta) = 0$ for any $\theta,\theta'\in\Theta$ such that $\theta' \ne \underline{r}$ and $\theta' \ne \bar r(\theta)$. Say this modification yields contract p_1. Obviously, $p_1(\bar r(\theta),\theta)$ is still the largest element in the column of p_1 indexed by θ. Consider $\forall\theta' \ne \underline{r}$, by our modification procedure, we have that

$$\begin{aligned}
&\sum_{\theta\in\Theta} P_\theta p_1(\theta',\theta) = \sum_{\theta\in\Theta} P_\theta \mathbb{I}\{\theta' = \bar r(\theta)\} p_0(\theta',\theta) \\
&\le \sum_{\theta\in\Theta} P_\theta p_0(\theta',\theta) \le \sum_{\theta\in\Theta} P_\theta p_0(\underline{r},\theta) = \sum_{\theta\in\Theta} P_\theta p_1(\underline{r},\theta),
\end{aligned}$$

thus $\underline{r}$ is still a prior optimal report under contract p_1. Besides, the invariance of the values of $\underline{r}$, $\{\bar{r}(\theta)\}_{\theta\in\Theta}$, $p_0(\bar{r}(\theta),\theta)$, $p_0(\underline{r},\theta)$ for any $\theta\in\Theta$ implies that the agent's expected net utility under each action a' is also invariant, i.e.,

$$\begin{aligned}&\sum_{\theta\in\Theta} P_\theta\Big[q_{a'}p_0(\bar{r}(\theta),\theta)+(1-q_{a'})p_0(\underline{r},\theta)\Big]-c_{a'}\\=&\sum_{\theta\in\Theta} P_\theta\Big[q_{a'}p_1(\bar{r}(\theta),\theta)+(1-q_{a'})p_1(\underline{r},\theta)\Big]-c_{a'},\quad \forall a'\in[n].\end{aligned}$$

Consequently, action a still maximizes the agent's expected net utility. Thus the contract p_1 still implements the agent strategy $\big(a,\{\bar{r}(\theta)\}_{\theta\in\Theta},\underline{r}\big)$, which yields the same principal utility as p_0 does.

The second modification aims to transform p_1 into a diagonal contract p_3 (Here, we merge step (ii) and step (iii) described in the proof sketch for the convenience of narration), defined by the following formulae:

$$\begin{aligned}p_3(\theta',\theta)&=0,\quad \forall\theta,\theta'\in\Theta,\quad s.t.\quad \theta\neq\theta',\\p_3(\theta,\theta)&=p_1\big(\bar{r}(\theta),\theta\big),\quad \forall\theta\notin\{\theta_0,\theta^*\},\\p_3(\theta_0,\theta_0)&=\frac{P_{\theta^*}}{P_{\theta_0}}p_1\big(\bar{r}(\theta^*),\theta^*\big),\\p_3(\theta^*,\theta^*)&=\frac{P_{\theta_0}}{P_{\theta^*}}p_1\big(\bar{r}(\theta_0),\theta_0\big),\end{aligned}$$

where the principal value determines

$$\theta_0\in\arg\max_{\theta'\in\Theta}\sum_{\theta\in\Theta}P_\theta v(\theta',\theta)$$

and θ^* is defined such that $\theta^*\in\arg\max_{\theta\in\Theta}P_\theta p_1\big(\bar{r}(\theta),\theta\big)$. Since p_3 is a diagonal contract, the agent truthfully reports when he knows the true state is θ. When the agent reports any $\theta'\neq\theta_0$, the expected payment

$$\begin{aligned}&\sum_{\theta\in\Theta}P_\theta p_3(\theta',\theta)=P_{\theta'}p_3(\theta',\theta')\\\leq&P_{\theta^*}p_1\big(\bar{r}(\theta^*),\theta^*\big)=P_{\theta_0}p_3(\theta_0,\theta_0)=\sum_{\theta\in\Theta}P_\theta p_3(\theta_0,\theta),\end{aligned}$$

which implies that θ_0 is an optimal report with respect to the prior distribution $\mathbb{P}_\theta$ under the contract p_3. We claim that there exists an action $a'\geq a$ that is incentivized by the contract p_3. Now, suppose this claim does not hold, then there exists $\tilde{a}<a$ such that $\forall a'\geq a$,

$$\begin{aligned}&q_{\tilde{a}}\sum_{\theta\in\Theta}P_\theta p_3(\theta,\theta)+(1-q_{\tilde{a}})\sum_{\theta\in\Theta}P_\theta p_3(\theta_0,\theta)-c_{\tilde{a}}\\>&q_{a'}\sum_{\theta\in\Theta}P_\theta p_3(\theta,\theta)+(1-q_{a'})\sum_{\theta\in\Theta}P_\theta p_3(\theta_0,\theta)-c_{a'}.\end{aligned}$$

Note that since the contract p_1 incentivizes action a, we have

$$\begin{aligned}&q_a\sum_{\theta\in\Theta}P_\theta p_1\big(\bar{r}(\theta),\theta\big)+(1-q_a)\sum_{\theta\in\Theta}P_\theta p_1(\underline{r},\theta)-c_a\\\geq&q_{\tilde{a}}\sum_{\theta\in\Theta}P_\theta p_1\big(\bar{r}(\theta),\theta\big)+(1-q_{\tilde{a}})\sum_{\theta\in\Theta}P_\theta p_1(\underline{r},\theta)-c_{\tilde{a}}.\end{aligned}$$

Besides, we notice that $\sum_{\theta\in\Theta}P_\theta p_1\big(\bar{r}(\theta),\theta\big)=\sum_{\theta\in\Theta}P_\theta p_3(\theta,\theta)$ by the construction of p_3 and that

$$\begin{aligned}&\sum_{\theta\in\Theta}P_\theta p_1(\underline{r},\theta)\geq\sum_{\theta\in\Theta}P_\theta p_1\big(\bar{r}(\theta^*),\theta\big)\geq P_{\theta^*}p_1\big(\bar{r}(\theta^*),\theta^*\big)\\=&P_{\theta_0}p_3(\theta_0,\theta_0)=\sum_{\theta\in\Theta}P_\theta p_3(\theta_0,\theta).\end{aligned}$$

These phenomena together imply that

$$\begin{aligned}&(q_a-q_{\tilde{a}})\sum_{\theta\in\Theta}P_\theta p_1\big(\bar{r}(\theta),\theta\big)+(q_{\tilde{a}}-q_a)\sum_{\theta\in\Theta}P_\theta p_1(\underline{r},\theta)+c_{\tilde{a}}-c_a\geq 0\\>&(q_a-q_{\tilde{a}})\sum_{\theta\in\Theta}P_\theta p_3(\theta,\theta)+(q_{\tilde{a}}-q_a)\sum_{\theta\in\Theta}P_\theta p_3(\theta_0,\theta)+c_{\tilde{a}}-c_a\\\geq&(q_a-q_{\tilde{a}})\sum_{\theta\in\Theta}P_\theta p_1\big(\bar{r}(\theta),\theta\big)+(q_{\tilde{a}}-q_a)\sum_{\theta\in\Theta}P_\theta p_1(\underline{r},\theta)+c_{\tilde{a}}-c_a,\end{aligned}$$

which is a contradiction. Thus we have justified our previous claim. Now we consider the principal's expected net utility under the contract p_3 (the action $a' \geq a$ is incentivized by p_3)

$$\begin{aligned}
&\sum_{\theta\in\Theta} P_\theta q_{a'}\big(v(\theta,\theta) - p_3(\theta,\theta)\big) + \sum_{\theta\in\Theta} P_\theta(1-q_{a'})\big(v(\theta_0,\theta) - p_3(\theta_0,\theta)\big) \\
\geq &\sum_{\theta\in\Theta} P_\theta q_{a'}\big(v(\theta,\theta) - p_1(\bar{r}(\theta),\theta)\big) + \sum_{\theta\in\Theta} P_\theta(1-q_{a'})\big(v(\theta_0,\theta) - p_1(\underline{r},\theta)\big) \\
\geq &\sum_{\theta\in\Theta} P_\theta q_{a'}\big(v(\theta,\theta) - p_1(\bar{r}(\theta),\theta)\big) + \sum_{\theta\in\Theta} P_\theta(1-q_{a'})\big(v(\underline{r},\theta) - p_1(\underline{r},\theta)\big).
\end{aligned}$$

Without loss of generality, we can assume that

$$\sum_{\theta\in\Theta} P_\theta\big(v(\theta,\theta) - p_1(\bar{r}(\theta),\theta)\big) > \sum_{\theta\in\Theta} P_\theta\big(v(\underline{r},\theta) - p_1(\underline{r},\theta)\big) \tag{14}$$

since otherwise the principal's expected net utility under the contract p_1 can be simply (weakly) improved by replacing p_1 with the naive zero-payment diagonal contract $p_1' = 0$:

$$\begin{aligned}
&\sum_{\theta\in\Theta} P_\theta q_a\big(v(\theta,\theta) - p_1(\bar{r}(\theta),\theta)\big) + \sum_{\theta\in\Theta} P_\theta(1-q_a)\big(v(\underline{r},\theta) - p_1(\underline{r},\theta)\big) \\
\leq &\sum_{\theta\in\Theta} P_\theta\big(v(\underline{r},\theta) - p_1(\underline{r},\theta)\big) \leq \sum_{\theta\in\Theta} P_\theta\big(v(\underline{r},\theta) - p_1'(\underline{r},\theta)\big) \\
= &\sum_{\theta\in\Theta} P_\theta v(\underline{r},\theta).
\end{aligned}$$

Given assumption (14), we have that

$$\begin{aligned}
&\sum_{\theta\in\Theta} P_\theta q_{a'}\big(v(\theta,\theta) - p_3(\theta,\theta)\big) + \sum_{\theta\in\Theta} P_\theta(1-q_{a'})\big(v(\theta_0,\theta) - p_3(\theta_0,\theta)\big) \\
\geq &\sum_{\theta\in\Theta} P_\theta q_a\big(v(\theta,\theta) - p_1(\bar{r}(\theta),\theta)\big) + \sum_{\theta\in\Theta} P_\theta(1-q_a)\big(v(\underline{r},\theta) - p_1(\underline{r},\theta)\big).
\end{aligned}$$

As a result, p_3 is a non-negative diagonal contract that yields (weakly) higher expected net utility for the principal than the original contract p_0 does.

□

Proof of Theorem 2. By Lemma 1, solving the program P1 does not require enumerating all possible realizations of $\big(a, \{\bar{r}(\theta)\}_{\theta\in\Theta}, \underline{r}\big)$. It is sufficient to consider all $n \cdot m$ combinations of $(a, \underline{r})$, while setting $\bar{r}(\theta) = \theta, \forall\theta \in \Theta$ and $p(\theta',\theta) = 0, \forall\theta',\theta \in \Theta$ such that $\theta' \neq \theta$. For each realization of $(a, \underline{r})$, the program P1 becomes the following linear program

$$\begin{aligned}
\max_{\mathrm{diag}(p)} \quad & \sum_{\theta\in\Theta} P_\theta q_a\big(v(\theta,\theta) - p(\theta,\theta)\big) + \sum_{\theta\in\Theta} P_\theta(1-q_a)v(\underline{r},\theta) - (1-q_a)P_{\underline{r}}p(\underline{r},\underline{r}) \\
s.t. \quad & P_{\underline{r}}p(\underline{r},\underline{r}) \geq P_{\theta'}p(\theta',\theta'), \quad \forall\theta' \in \Theta, \\
& \sum_{\theta\in\Theta} P_\theta q_a p(\theta,\theta) + P_{\underline{r}}(1-q_a)p(\underline{r},\underline{r}) - c_a \geq \\
& \quad \sum_{\theta\in\Theta} P_\theta q_{a'} p(\theta,\theta) + P_{\underline{r}}(1-q_{a'})p(\underline{r},\underline{r}) - c_{a'}, \forall a' \in [n] \\
& p(\theta,\theta) \geq 0, \quad \forall\theta \in \Theta,
\end{aligned}$$

where $\mathrm{diag}(p)$ is the vector of the diagonal elements of the matrix p. Since the variables in this linear program are only $\{p(\theta,\theta)\}_{\theta\in\Theta}$, the objective function is equivalent to $-\sum_{\theta\in\Theta} P_\theta q_a p(\theta,\theta) - (1-q_a)P_{\underline{r}}p(\underline{r},\underline{r})$ as the constant term in the objective function does not affect the solution of a linear program. Thus, it can be checked that this linear program is equal to that in Algorithm 1. Algorithm 1 searches over all possible pairs of $(a, \underline{r})$ and, for the pair that maximizes the objective function of P1, outputs the diagonal contract p that implements this pair and achieves the corresponding objective value, ensuring that p is an optimal solution to P1. □

B MISSING PROOFS FOR SECTION 4.1 AND 4.2

Proof of Proposition 2. For any $(s, r) \in \Delta_{[2]} \times \mathcal{R}$, $u_A(s, r)$ can be expanded as

$$\begin{aligned}
u_A(s,r) =& \sum_{\theta\in\Theta} P_\theta\left[s_1 \sum_{\hat\theta\in\Theta} r_{\hat\theta}(1,\theta)\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} + s_2\Big(\sum_{\hat\theta\in\Theta} r_{\hat\theta}(2,\theta)\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} - c\Big)\right]\\
=& s_1 \sum_{\hat\theta\in\Theta} r_{\hat\theta}(1)\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} + s_2\Big(\sum_{\theta\in\Theta} P_\theta \sum_{\hat\theta\in\Theta} r_{\hat\theta}(2,\theta)\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} - c\Big)\\
\leq& s_1 \max_{\hat\theta\in\Theta}\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} + s_2\Big(\sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta}\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} - c\Big)\\
\leq& \max\left\{\max_{\hat\theta\in\Theta}\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta}, \sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta}\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} - c\right\},
\end{aligned}$$

where we wrote $r(1) = r(1, \theta), \forall \theta \in \Theta$ by the definition of $\mathcal{R}$. Let (s^1, r^1) represent the agent strategy defined in (7), (8) and (9). It can be validated that

$$\sum_{\hat\theta\in\Theta} r^1_{\hat\theta}(2,\theta)\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} = \int_{\mathbb{R}} p(\bar r(\theta),x)dF_{X|\bar r(\theta),\theta} = \max_{\hat\theta\in\Theta}\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta}, \forall\theta\in\Theta,$$

$$\sum_{\theta\in\Theta} P_\theta \sum_{\hat\theta\in\Theta} r^1_{\hat\theta}(1,\theta)\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} = \sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\underline{r},x)dF_{X|\underline{r},\theta} = \max_{\hat\theta\in\Theta}\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta},$$

and furthermore,

$$\begin{aligned}
u_A(s^1,r^1) =& s^1_1 \max_{\hat\theta\in\Theta}\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} + s^1_2\Big(\sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta}\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} - c\Big)\\
=& \max\left\{\max_{\hat\theta\in\Theta}\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta}, \sum_{\theta\in\Theta} P_\theta \max_{\hat\theta\in\Theta}\int_{\mathbb{R}} p(\hat\theta,x)dF_{X|\hat\theta,\theta} - c\right\}\\
\geq& \max_{(s,r)\in\Delta_{[2]}\times\mathcal{R}} u_A(s,r)
\end{aligned}$$

by our upper bound derived at the beginning of this proof. □

Proof of Proposition 3. Say the principal adopts a contract p that incentivizes positive agent effort and truth-telling. Then, by the definition of these constraints, p satisfies

$$\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} - c \geq \max_{\theta'\in\Theta}\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta}, \tag{15}$$

$$\int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} \geq \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta}, \quad \forall\theta,\theta'\in\Theta. \tag{16}$$

The expected payment is given by $\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta}$. We define a vector $\vec{y}$ (each element of which is indexed by a $\theta \in \Theta$) such that

$$y_\theta = P_\theta \int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta}, \quad \forall\theta\in\Theta.$$

Now, we show that $\vec{y}$ is a feasible solution to the linear program L1. First of all, we know $\vec{y} \geq 0$. Secondly, we have that for any $\theta' \in \Theta$,

$$\begin{aligned}
&\sum_{\theta\in\Theta} y_\theta - c\\
\geq& \sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta} = \int_{\mathbb{R}} p(\theta',x)\sum_{\theta\in\Theta} P_\theta\phi_\eta\big(x - v(\theta',\theta)\big)dx\\
\geq& \int_{\mathbb{R}} p(\theta',x)\inf_{s\in\mathbb{R}}\sum_{\theta\in\Theta}\alpha_{\theta,\theta'}(s)\cdot P_{\theta'}\phi_\eta\big(x - v(\theta',\theta')\big)dx\\
=& \inf_{s\in\mathbb{R}}\sum_{\theta\in\Theta}\alpha_{\theta,\theta'}(s)\cdot P_{\theta'}\int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta'} = \Big[\inf_{s\in\mathbb{R}}\sum_{\theta\in\Theta}\alpha_{\theta,\theta'}(s)\Big]y_{\theta'},
\end{aligned}$$

where the first inequality is by (15) and the second inequality is by the definition of $\alpha_{\theta,\theta'}(s)$. Lastly, we have that for any $\theta, \theta' \in \Theta$,

$$\begin{aligned} y_\theta \geq & P_\theta \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta} = P_\theta \int_{\mathbb{R}} p(\theta', x) \phi_\eta\big(x - v(\theta', \theta)\big) dx \\ \geq & \int_{\mathbb{R}} p(\theta', x) \inf_{s\in\mathbb{R}} \alpha_{\theta,\theta'}(s) \cdot P_{\theta'} \phi_\eta\big(x - v(\theta', \theta')\big) dx \\ = & \inf_{s\in\mathbb{R}} \alpha_{\theta,\theta'}(s) \cdot P_{\theta'} \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta'} = \Big[\inf_{s\in\mathbb{R}} \alpha_{\theta,\theta'}(s)\Big] y_{\theta'}, \end{aligned}$$

where the first inequality is by (16) and the second inequality is also by the definition of $\alpha_{\theta,\theta'}(s)$. We have validated that $\vec{y}$ is a feasible solution to L1 since all its constraints (10), (11) and (12) are satisfied by $\vec{y}$. Let $\vec{x}^*$ be an optimal solution to L1. We notice that the expected payment of contract p

$$\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta, x) dF_{X|\theta,\theta} = \sum_{\theta\in\Theta} y_\theta \geq \sum_{\theta\in\Theta} x^*_\theta,$$

since $\vec{x}^*$ optimizes the objective value of L1. Thus, we have shown that $\sum_{\theta\in\Theta} x^*_\theta$ is a lower bound for the expected payment of any contract that incentivizes positive agent effort and truth-telling. □

Proof of Lemma 2. Assume that $\vec{t}^*$ is an optimal solution to O1. Now, we are trying to construct a solution to O2 which is as close to $\vec{t}^*$ as possible. We know that $a_{\theta,\theta'} t^*_{\theta'} \leq t^*_\theta, \forall \theta \neq \theta'$ but there can be θ, θ' such that $a'_{\theta,\theta'} t^*_{\theta'} > t^*_\theta$. The first step of our proof is to reweight $\vec{t}^*$ and construct an intermediate solution $\vec{y}$ such that the stronger constraints $a'_{\theta,\theta'} y_{\theta'} \leq y_\theta, \forall \theta \neq \theta'$ hold. By our assumption,

$$a_{\theta,\theta'} \leq a'_{\theta,\theta'} \leq \frac{P_\theta}{P_{\theta'}}, \quad \forall \theta, \theta' \in \Theta, \theta \neq \theta'.$$

This inspires us to define $t'_\theta = P_\theta \sum_{\theta'\in\Theta} t^*_{\theta'}, \forall \theta \in \Theta$. The definition of $\vec{t}'$ implies that

$$a'_{\theta,\theta'} t'_{\theta'} = a'_{\theta,\theta'} t'_\theta \frac{P_{\theta'}}{P_\theta} \leq t'_\theta, \quad \forall \theta, \theta' \in \Theta, \theta \neq \theta',$$

and $\sum_{\theta\in\Theta} t'_\theta = \sum_{\theta\in\Theta} t^*_\theta$. Furthermore, for a constant $\lambda \in (0, 1)$, we construct the intermediate solution $\vec{y}$ such that $y_\theta = \lambda t^*_\theta + (1-\lambda) t'_\theta, \forall \theta \in \Theta$. Specifically, we set

$$\lambda = \min_{\theta\neq\theta' : a'_{\theta,\theta'} t^*_{\theta'} > t^*_\theta} \frac{t'_\theta - a'_{\theta,\theta'} t'_{\theta'}}{t'_\theta - a'_{\theta,\theta'} t'_{\theta'} + \epsilon_{\theta,\theta'} t^*_{\theta'}}.$$

Fix any $\theta \neq \theta'$, we now attempt to show that $\vec{y}$ satisfies $a'_{\theta,\theta'} y_{\theta'} \leq y_\theta, \forall \theta \neq \theta'$. There are two possibilities:
(1) $a'_{\theta,\theta'} t^*_{\theta'} \leq t^*_\theta$. Then trivially,

$$\begin{aligned} a'_{\theta,\theta'} y_{\theta'} &= a'_{\theta,\theta'} \big[\lambda t^*_{\theta'} + (1-\lambda) t'_{\theta'}\big] \\ &= \lambda a'_{\theta,\theta'} t^*_{\theta'} + (1-\lambda) a'_{\theta,\theta'} t'_{\theta'} \\ &\leq \lambda t^*_\theta + (1-\lambda) t'_\theta = y_\theta \end{aligned}$$

by the definition of $\vec{t}^*, \vec{t}'$.
(2) $a'_{\theta,\theta'} t^*_{\theta'} > t^*_\theta$. We have that

$$\begin{aligned} a'_{\theta,\theta'} y_{\theta'} &= a'_{\theta,\theta'} \big[\lambda t^*_{\theta'} + (1-\lambda) t'_{\theta'}\big] \\ &= a'_{\theta,\theta'} \big[\lambda t^*_{\theta'} + (1-\lambda) t'_{\theta'}\big] \\ &= \lambda \big[a'_{\theta,\theta'} t^*_{\theta'} - t^*_\theta\big] + \lambda t^*_\theta + (1-\lambda) a'_{\theta,\theta'} t'_{\theta'}. \end{aligned}$$

By the definition of λ, we have that

$$\begin{aligned} \lambda \leq & \frac{t'_\theta - a'_{\theta,\theta'} t'_{\theta'}}{t'_\theta - a'_{\theta,\theta'} t'_{\theta'} + \epsilon_{\theta,\theta'} t^*_{\theta'}} \\ \leq & \frac{t'_\theta - a'_{\theta,\theta'} t'_{\theta'}}{t'_\theta - a'_{\theta,\theta'} t'_{\theta'} + (a'_{\theta,\theta'} - a_{\theta,\theta'}) t^*_{\theta'}} \\ \leq & \frac{t'_\theta - a'_{\theta,\theta'} t'_{\theta'}}{t'_\theta - a'_{\theta,\theta'} t'_{\theta'} + a'_{\theta,\theta'} t^*_{\theta'} - t^*_\theta}, \end{aligned}$$

which directly implies

$$\lambda\left[a'_{\theta,\theta'}t^*_{\theta'} - t^*_\theta\right] \le (1-\lambda)\left[t'_\theta - a'_{\theta,\theta'}t'_{\theta'}\right].$$

Returning to our derivation regarding $a'_{\theta,\theta'}y_{\theta'}$, we have that

$$\begin{aligned} a'_{\theta,\theta'}y_{\theta'} &\le (1-\lambda)\left[t'_\theta - a'_{\theta,\theta'}t'_{\theta'}\right] + \lambda t^*_\theta + (1-\lambda)a'_{\theta,\theta'}t'_{\theta'} \\ &= (1-\lambda)t'_\theta + \lambda t^*_\theta = y_\theta. \end{aligned}$$

As a result, we have shown that the intermediate solution $\vec{y}$ satisfies $a'_{\theta,\theta'}y_{\theta'} \le y_\theta, \forall \theta \ne \theta'$. We also mention here that

$$\begin{aligned} \sum_{\theta\in\Theta} y_\theta &= \lambda \sum_{\theta\in\Theta} t^*_\theta + (1-\lambda)\sum_{\theta\in\Theta} t'_\theta \\ &= \lambda \sum_{\theta\in\Theta} t^*_\theta + (1-\lambda)\sum_{\theta\in\Theta} t^*_\theta = \sum_{\theta\in\Theta} t^*_\theta. \end{aligned}$$

We now continue to the second step. We attempt to find a constant $\mu > 1$, which is as close to 1 as possible, such that $\mu\vec{y}$ is a feasible solution to the second linear program O2. The most natural choice of μ is

$$\mu = \max_{\theta'\in\Theta} \frac{c}{\sum_{\theta\in\Theta} y_\theta - b'_{\theta'}y_{\theta'}},$$

since it is the smallest μ such that all the positive agent effort incentive compatibility constraints are satisfied. To proceed, we make the following claim:

$$\max_{\theta'\in\Theta} b_{\theta'}t^*_{\theta'} + c = \sum_{\theta\in\Theta} t^*_\theta.$$

Suppose this formula does not hold. Then we have that $\max_{\theta'\in\Theta} b_{\theta'}t^*_{\theta'} + c < \sum_{\theta\in\Theta} t^*_\theta$. It can be verified that the vector $\vec{t}$ defined such that

$$t_\theta = t^*_\theta \cdot \frac{c}{\sum_{\theta\in\Theta} t^*_\theta - \max_{\theta'\in\Theta} b_{\theta'}t^*_{\theta'}} < t^*_\theta, \quad \forall \theta \in \Theta$$

is a feasible solution to the first linear program O1 with a strictly lower objective value, which is contradictory to the fact that $\vec{t}^*$ is the optimal solution to O1.

We return to μ and notice that, by our discussion above, it can be rewritten as

$$\begin{aligned} \mu &= \frac{c}{\sum_{\theta\in\Theta} y_\theta - \max_{\theta'\in\Theta} b'_{\theta'}y_{\theta'}} \\ &= \frac{c}{\sum_{\theta\in\Theta} t^*_\theta - \max_{\theta'\in\Theta} b'_{\theta'}y_{\theta'}} \\ &= \frac{1}{1 - \left[\max_{\theta'\in\Theta} b'_{\theta'}y_{\theta'} - \max_{\theta'\in\Theta} b_{\theta'}t^*_{\theta'}\right]/c} \\ &\le \frac{1}{1 - \max_{\theta'\in\Theta}\left[b'_{\theta'}y_{\theta'} - b_{\theta'}t^*_{\theta'}\right]/c}. \end{aligned}$$

Fix any $\theta' \in [m]$, we focus on $b'_{\theta'}y_{\theta'} - b_{\theta'}t^*_{\theta'}$ alone and show that

$$\begin{aligned} & b'_{\theta'}y_{\theta'} - b_{\theta'}t^*_{\theta'} \\ =& b'_{\theta'}\left[\lambda t^*_{\theta'} + (1-\lambda)t'_{\theta'}\right] - b_{\theta'}t^*_{\theta'} \\ =& (b'_{\theta'} - b_{\theta'})t^*_{\theta'} + (1-\lambda)b'_{\theta'}(t'_{\theta'} - t^*_{\theta'}) \\ \le& \kappa_{\theta'}t^*_{\theta'} + (1-\lambda)b'_{\theta'}(P_{\theta'}\sum_{\theta\in\Theta} t^*_\theta - t^*_{\theta'}) \\ \le& \kappa_{\theta'}t^*_{\theta'} + (1-\lambda)b'_{\theta'}P_{\theta'}\sum_{\theta\in\Theta} t^*_\theta \\ \le& \kappa_{\theta'}t^*_{\theta'} + (1-\lambda)\sum_{\theta\in\Theta} t^*_\theta. \end{aligned}$$

To continue our derivation, we still need upper bounds for $1-\lambda$ and OPT. Now, we turn to consider $1-\lambda$:

$$\begin{aligned}1-\lambda &= \max_{\theta\neq\theta':a'_{\theta,\theta'}t^*_{\theta'}>t^*_\theta} \frac{\epsilon_{\theta,\theta'}t^*_{\theta'}}{t'_\theta - a'_{\theta,\theta'}t'_{\theta'} + \epsilon_{\theta,\theta'}t^*_{\theta'}} \\ &= \max_{\theta\neq\theta':a'_{\theta,\theta'}t^*_{\theta'}>t^*_\theta} \frac{\epsilon_{\theta,\theta'}t^*_{\theta'}}{[P_\theta - a'_{\theta,\theta'}P_{\theta'}]\sum_{\tilde\theta\in\Theta}t^*_{\tilde\theta} + \epsilon_{\theta,\theta'}t^*_{\theta'}} \\ &\leq \max_{\theta\neq\theta':a'_{\theta,\theta'}t^*_{\theta'}>t^*_\theta} \frac{\epsilon_{\theta,\theta'}}{[P_\theta - a'_{\theta,\theta'}P_{\theta'}] + \epsilon_{\theta,\theta'}} \\ &\leq \max_{\theta\neq\theta'} \frac{\epsilon_{\theta,\theta'}}{[P_\theta - a'_{\theta,\theta'}P_{\theta'}] + \epsilon_{\theta,\theta'}}.\end{aligned}$$

To obtain a valid upper bound for the value of OPT, we notice that although $\vec{t}'$ may not be a feasible solution to O1, there must be a constant scaling factor $s \geq 1$ such that $s\vec{t}'$ becomes a feasible solution. The smallest possible s is determined by the following equation,

$$\max_{\theta'\in\Theta} b_{\theta'}st'_{\theta'} + c = \sum_{\theta\in\Theta} st'_\theta,$$

which implies a valid upper bound for OPT:

$$\begin{aligned}\text{OPT} &\leq \sum_{\theta\in\Theta} st'_\theta \\ &= \frac{c\sum_{\theta\in\Theta}t'_\theta}{\sum_{\theta\in\Theta}t'_\theta - \max_{\theta'\in\Theta}b_{\theta'}t'_{\theta'}} \\ &= \frac{c}{1-\max_{\theta'\in\Theta}b_{\theta'}P_{\theta'}}.\end{aligned}$$

As a result, we have that

$$\begin{aligned}\text{OPT}' &\leq \sum_{\theta\in\Theta}\mu y_\theta \\ &\leq \frac{\sum_{\theta\in\Theta}t^*_\theta}{1-\max_{\theta'\in\Theta}\left[b'_{\theta'}y_{\theta'} - b_{\theta'}t^*_{\theta'}\right]/c} \\ &\leq \frac{\text{OPT}}{1-\frac{1}{c}\max_{\theta\in\Theta}[\kappa_\theta t^*_\theta + (1-\lambda)\text{OPT}]} \\ &\leq \frac{\text{OPT}}{1-\frac{1}{c}\max_{\theta\in\Theta}[\kappa_\theta + (1-\lambda)]\text{OPT}} \\ &\leq \frac{\text{OPT}}{1-\frac{\max_{\theta\in\Theta}[\kappa_\theta+(1-\lambda)]}{1-\max_{\theta\in\Theta}b_\theta P_\theta}},\end{aligned}$$

which completes the proof. □

Proof of Theorem 5. Note that in the contract framework defined in Algorithm 2 the payment radius $\rho_\theta, \forall\theta\in\Theta$ are regarded as inputs. Now we focus on a realization of these parameters and analyze the performance of this specific contract. We set

$$\rho_\theta = F_\eta^{-1}\left[\frac{1}{2} + \frac{cP_\theta^{-1}B^{-1}}{1-\max_{\theta'\in\Theta}\sum_{\tilde\theta\in\Theta}P_{\tilde\theta}\frac{\phi_\eta(v(\theta',\theta')-v(\theta',\tilde\theta))}{\phi_\eta(0)}}\right], \quad \forall\theta\in\Theta.$$

To make this contract work, we need a sufficiently large budget constraint B. Specifically, these are our assumptions on B: $\forall\theta\in\Theta$,

(1) $B \geq \frac{12\phi_\eta(0)Lc}{[\phi_\eta(0)-\phi_\eta(\underline{l})]^3\underline{P}^2}$.
(2) $\rho_\theta \leq \frac{\phi_\eta(0)}{3L}\left[1-\sum_{\theta'\in\Theta}P_{\theta'}\frac{\phi_\eta(v(\theta,\theta)-v(\theta,\theta'))}{\phi_\eta(0)}\right]$.
(3) $\rho_\theta \leq \phi_\eta(0)/L$.
(4) $\rho_\theta \leq \bar{l}$.

Throughout this proof, define $b_\theta = \sum_{\theta'\in\Theta} \frac{P_{\theta'}}{P_\theta} \frac{\phi_\eta(v(\theta,\theta)-v(\theta,\theta'))}{\phi_\eta(0)}$. When these assumptions hold, we can derive an upper bound of ρ_θ w.r.t B. By the definition of ρ_θ, we have that

$$\begin{aligned}
&\int_0^{\rho_\theta} dF_\eta = \frac{c}{1-\max_{\theta'\in\Theta} b_{\theta'}P_{\theta'}} \frac{1}{BP_\theta} \\
=&\int_{-\rho_\theta}^{0} \phi_\eta(x)dx \geq \int_{-\rho_\theta}^{0} [\phi_\eta(0)+Lx]dx \\
\geq&\rho_\theta\phi_\eta(0) - \frac{1}{2}L\rho_\theta^2 \geq \frac{1}{2}\rho_\theta\phi_\eta(0),
\end{aligned}$$

and furthermore

$$\rho_\theta \leq \frac{2c}{1-\max_{\theta'\in\Theta} b_{\theta'}P_{\theta'}} \frac{1}{BP_\theta\phi_\eta(0)}. \tag{17}$$

In the below, we attempt to show that the contract p computed by Algorithm 2 (if $\sum_{\theta\in\Theta} z_\theta \leq \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \underline{u}_P$) has the following properties:

(1) The budget constraint is never violated: $B_\theta \leq B, \forall\theta\in\Theta$.
(2) p incentivizes positive agent effort and truth-telling simultaneously.
(3) The expected principal utility induced by p is very close to the optimal utility any truth-telling incentivizing contract can achieve.

Now, we show the first property holds. For any $\theta\in\Theta$, by the definition of B_θ, we have that

$$\begin{aligned}
B_\theta =& \frac{z_\theta}{P_\theta \int_{-\rho_\theta}^{\rho_\theta} dF_\eta} \leq \frac{\sum_{\theta'\in\Theta} z_{\theta'}}{P_\theta \int_{-\rho_\theta}^{\rho_\theta} dF_\eta} \\
\leq& \frac{1}{P_\theta \int_{-\rho_\theta}^{\rho_\theta} dF_\eta} \frac{c}{1-\max_{\theta'\in\Theta} b'_{\theta'}P_{\theta'}} \\
\leq& \frac{1}{P_\theta \int_{-\rho_\theta}^{\rho_\theta} dF_\eta} \frac{c}{1-\max_{\theta'\in\Theta} \frac{1}{2}[P_{\theta'}^{-1}+b_{\theta'}]P_{\theta'}} \\
=& \frac{1}{P_\theta \int_{-\rho_\theta}^{\rho_\theta} dF_\eta} \frac{2c}{1-\max_{\theta'\in\Theta} b_{\theta'}P_{\theta'}} = B.
\end{aligned}$$

We elaborate more on this derivation. The first inequality is by $z_\theta \geq 0, \forall\theta\in\Theta$ due to the last constraint of the linear program in Algorithm 2. The second inequality is obtained similarly as we did in the proof of Lemma 2 to derive a simple upper bound for OPT. The third inequality is from our assumption on B. The last equality is from the chosen definition of ρ_θ.

We turn to the second property. Firstly, we validate the truth-telling constraints. For any $\theta,\theta'\in\Theta$, by our construction of the contract p,

$$\begin{aligned}
&\int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} \\
=&\int_{v(\theta,\theta)-\rho_\theta}^{v(\theta,\theta)+\rho_\theta} B_\theta\phi_\eta\big(x-v(\theta,\theta)\big)dx \\
=&\int_{-\rho_\theta}^{\rho_\theta} B_\theta dF_\eta = \frac{z_\theta}{P_\theta},
\end{aligned}$$

while

$$\begin{aligned}
&\int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta} = \int_{v(\theta',\theta')-\rho_{\theta'}}^{v(\theta',\theta')+\rho_{\theta'}} B_{\theta'}\phi_\eta\big(x-v(\theta',\theta)\big)dx \\
=&\frac{\int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} B_{\theta'}dF_\eta}{\int_{-\rho_{\theta'}}^{\rho_{\theta'}} B_{\theta'}dF_\eta} \cdot \int_{-\rho_{\theta'}}^{\rho_{\theta'}} B_{\theta'}dF_\eta \\
=&\frac{\int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} dF_\eta}{\int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \cdot \frac{z_{\theta'}}{P_{\theta'}}.
\end{aligned}$$

Thus we have

$$\int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} \geq \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta}$$

since $\vec{z}$ is a feasible solution to the program in Algorithm 2. Secondly, we validate the positive agent effort constraints. For any $\theta' \in \Theta$, by our construction of the contract p, similarly,

$$\begin{aligned}
&\sum_{\theta \in \Theta} P_\theta \int_{\mathbb{R}} p(\theta, x) dF_{X|\theta,\theta} - c = \sum_{\theta \in \Theta} z_\theta - c \\
&\geq \left[\sum_{\theta \in \Theta} \frac{P_\theta \int_{v(\theta',\theta') - v(\theta',\theta) - \rho_{\theta'}}^{v(\theta',\theta') - v(\theta',\theta) + \rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \right] z_{\theta'} \\
&= \sum_{\theta \in \Theta} P_\theta \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta}.
\end{aligned}$$

Thirdly, we analyze the expected payment of the contract. Before that, we show that the constraints of the linear program in Algorithm 2 are (at least weakly) tighter than those of the linear program L1 in Proposition 3. For any $\theta, \theta' \in \Theta$ and $x \in \mathbb{R}$, we notice that

$$\frac{\phi_\eta\big(x + v(\theta', \theta') - v(\theta', \theta)\big)}{\phi_\eta(x)} \geq \inf_{s \in \mathbb{R}} \frac{\phi_\eta\big(s - v(\theta', \theta)\big)}{\phi_\eta\big(s - v(\theta', \theta')\big)}$$

and

$$\begin{aligned}
&\sum_{\theta \in \Theta} \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta\big(x + v(\theta', \theta') - v(\theta', \theta)\big)}{\phi_\eta(x)} \\
&\geq \inf_{s \in \mathbb{R}} \sum_{\theta \in \Theta} \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta\big(s - v(\theta', \theta)\big)}{\phi_\eta\big(s - v(\theta', \theta')\big)}.
\end{aligned}$$

These two inequalities directly imply that

$$\begin{aligned}
&\frac{P_\theta \int_{v(\theta',\theta') - v(\theta',\theta) - \rho_{\theta'}}^{v(\theta',\theta') - v(\theta',\theta) + \rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \\
&= \frac{P_\theta \int_{-\rho_{\theta'}}^{\rho_{\theta'}} \phi_\eta\big(x + v(\theta', \theta') - v(\theta', \theta)\big) dx}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \\
&\geq \inf_{s \in \mathbb{R}} \frac{\phi_\eta\big(s - v(\theta', \theta)\big)}{\phi_\eta\big(s - v(\theta', \theta')\big)} \cdot \frac{P_\theta \int_{-\rho_{\theta'}}^{\rho_{\theta'}} \phi_\eta(x) dx}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \\
&= \inf_{s \in \mathbb{R}} \alpha_{\theta,\theta'}(s)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\theta \in \Theta} \frac{P_\theta \int_{v(\theta',\theta') - v(\theta',\theta) - \rho_{\theta'}}^{v(\theta',\theta') - v(\theta',\theta) + \rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \\
&= \sum_{\theta \in \Theta} \frac{P_\theta \int_{-\rho_{\theta'}}^{\rho_{\theta'}} \phi_\eta\big(x + v(\theta', \theta') - v(\theta', \theta)\big) dx}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \\
&= \frac{\int_{-\rho_{\theta'}}^{\rho_{\theta'}} \sum_{\theta \in \Theta} \frac{P_\theta}{P_{\theta'}} \phi_\eta\big(x + v(\theta', \theta') - v(\theta', \theta)\big) dx}{\int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \\
&\geq \inf_{s \in \mathbb{R}} \sum_{\theta \in \Theta} \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta\big(s - v(\theta', \theta)\big)}{\phi_\eta\big(s - v(\theta', \theta')\big)} = \inf_{s \in \mathbb{R}} \sum_{\theta \in \Theta} \alpha_{\theta,\theta'}(s).
\end{aligned}$$

Assumption 3 on the noise distribution implies that

$$\begin{aligned}
\inf_{s \in \mathbb{R}} \alpha_{\theta,\theta'}(s) &= \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta\big(v(\theta', \theta') - v(\theta', \theta)\big)}{\phi_\eta(0)}, \\
\inf_{s \in \mathbb{R}} \sum_{\theta \in \Theta} \alpha_{\theta,\theta'}(s) &= \sum_{\theta \in \Theta} \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta\big(v(\theta', \theta') - v(\theta', \theta)\big)}{\phi_\eta(0)}.
\end{aligned}$$

To match the notation used in Lemma 2, for any $\theta, \theta' \in \Theta$ we define

$$\begin{cases} a_{\theta,\theta'} &= \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta\left(v(\theta',\theta')-v(\theta',\theta)\right)}{\phi_\eta(0)}, \\ a'_{\theta,\theta'} &= \frac{P_\theta \int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta}, \\ b_{\theta'} &= \sum_{\theta\in\Theta} \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta\left(v(\theta',\theta')-v(\theta',\theta)\right)}{\phi_\eta(0)}, \\ b'_{\theta'} &= \sum_{\theta\in\Theta} \frac{P_\theta \int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta}. \end{cases}$$

Defining $l_{\theta,\theta'} = v(\theta',\theta') - v(\theta',\theta), \forall \theta,\theta' \in \Theta$, we have that

$$\begin{aligned} & a'_{\theta,\theta'} - a_{\theta,\theta'} \\ =& \frac{P_\theta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} \left[\phi_\eta(x + l_{\theta,\theta'}) - \frac{\phi_\eta(l_{\theta,\theta'})}{\phi_\eta(0)}\phi_\eta(x)\right] dx \\ =& \frac{P_\theta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \left[\int_{-\rho_{\theta'}}^{0} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(x + |l_{\theta,\theta'}|)}{\phi_\eta(l_{\theta,\theta'})} dx \right. \\ & + \int_0^{\rho_{\theta'}} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(x + |l_{\theta,\theta'}|)}{\phi_\eta(l_{\theta,\theta'})} dx \\ & \left. - 2\int_0^{\rho_{\theta'}} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(x)}{\phi_\eta(0)} dx \right]. \end{aligned}$$

The Lipschitz continuity of ϕ_η implies that

$$\begin{aligned} & \int_{-\rho_{\theta'}}^{0} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(x + |l_{\theta,\theta'}|)}{\phi_\eta(l_{\theta,\theta'})} dx \\ \le & \int_{-\rho_{\theta'}}^{0} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(l_{\theta,\theta'}) - Lx}{\phi_\eta(l_{\theta,\theta'})} dx \\ =& \rho_{\theta'}\phi_\eta(l_{\theta,\theta'}) + \frac{L}{2}\rho_{\theta'}^2, \\ & \int_0^{\rho_{\theta'}} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(x + |l_{\theta,\theta'}|)}{\phi_\eta(l_{\theta,\theta'})} dx \\ \le & \int_0^{\rho_{\theta'}} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(l_{\theta,\theta'})}{\phi_\eta(l_{\theta,\theta'})} dx \\ =& \rho_{\theta'}\phi_\eta(l_{\theta,\theta'}), \end{aligned}$$

and

$$\begin{aligned} & 2\int_0^{\rho_{\theta'}} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(x)}{\phi_\eta(0)} dx \\ \ge & 2\int_0^{\rho_{\theta'}} \phi_\eta(l_{\theta,\theta'}) \frac{\phi_\eta(0) - Lx}{\phi_\eta(0)} dx \\ =& 2\rho_{\theta'}\phi_\eta(l_{\theta,\theta'}) - L\frac{\phi_\eta(l_{\theta,\theta'})}{\phi_\eta(0)}\rho_{\theta'}^2. \end{aligned}$$

As a consequence, we obtain that

$$\begin{aligned} a'_{\theta,\theta'} - a_{\theta,\theta'} \le & \frac{P_\theta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \left[\frac{L}{2}\rho_{\theta'}^2 + L\frac{\phi_\eta(l_{\theta,\theta'})}{\phi_\eta(0)}\rho_{\theta'}^2\right] \\ \le & \frac{P_\theta}{P_{\theta'}\left[2\rho_{\theta'}\phi_\eta(0) - L\rho_{\theta'}^2\right]} \frac{3L}{2}\rho_{\theta'}^2 \\ \le & \frac{3LP_\theta\rho_{\theta'}}{2P_{\theta'}\phi_\eta(0)} =: \epsilon_{\theta,\theta'}, \end{aligned}$$

where the second inequality is also by the Lipschitz assumption and the last inequality is by our assumption on B. We observe that

$$b'_{\theta'} - b_{\theta'} = \sum_{\theta \in \Theta} a'_{\theta,\theta'} - a_{\theta,\theta'} \leq \sum_{\theta \in \Theta} \epsilon_{\theta,\theta'},$$

so we define $\kappa_{\theta'} = \sum_{\theta \in \Theta} \epsilon_{\theta,\theta'}$.

Before deriving our final result, we still need to deal with the terms such as $1-\lambda$, $\max_{\theta \in \Theta} \kappa_\theta$ and $1 - \max_{\theta' \in \Theta} b_{\theta'} P_{\theta'}$. By Lemma 2,

$$\begin{aligned}
1 - \lambda &\leq \max_{\theta \neq \theta'} \frac{\epsilon_{\theta,\theta'}}{[P_\theta - a'_{\theta,\theta'} P_{\theta'}] + \epsilon_{\theta,\theta'}} \\
&\leq \max_{\theta \neq \theta'} \frac{\epsilon_{\theta,\theta'}}{[P_\theta - (a_{\theta,\theta'} + \epsilon_{\theta,\theta'}) P_{\theta'}] + \epsilon_{\theta,\theta'}} \\
&= \max_{\theta \neq \theta'} \frac{\epsilon_{\theta,\theta'}}{[P_\theta - a_{\theta,\theta'} P_{\theta'}] + (1 - P_{\theta'}) \epsilon_{\theta,\theta'}} \\
&\leq \max_{\theta \neq \theta'} \frac{\epsilon_{\theta,\theta'}}{P_\theta - a_{\theta,\theta'} P_{\theta'}}.
\end{aligned}$$

Note that by the definition of $a_{\theta,\theta'}$,

$$a_{\theta,\theta'} \leq \frac{P_\theta}{P_{\theta'}} \frac{\phi_\eta(\underline{l}_{\theta'})}{\phi_\eta(0)},$$

where we define $\underline{l}_\theta = \min_{\tilde{\theta} \neq \theta} |v(\theta,\theta) - v(\theta,\tilde{\theta})|, \forall \theta \in \Theta$. Thus $1 - \lambda$ can be further bounded as

$$\begin{aligned}
1 - \lambda &\leq \max_{\theta \neq \theta'} \frac{3 L P_\theta \rho_{\theta'}}{[P_\theta - a_{\theta,\theta'} P_{\theta'}] 2 P_{\theta'} \phi_\eta(0)} \\
&\leq \max_{\theta \in \Theta} \frac{3 L \rho_\theta}{[\phi_\eta(0) - \phi_\eta(\underline{l}_\theta)] 2 P_\theta} \\
&\leq \frac{3L}{2[\phi_\eta(0) - \phi_\eta(\underline{l})]} \max_{\theta \in \Theta} \frac{\rho_\theta}{P_\theta},
\end{aligned}$$

where $\underline{l} := \min_{\theta \in \Theta} \underline{l}_\theta$. By its definition,

$$\max_{\theta \in \Theta} \kappa_\theta = \frac{3L}{2\phi_\eta(0)} \max_{\theta \in \Theta} \frac{\rho_\theta}{P_\theta}.$$

Moreover,

$$\frac{1}{1 - \max_{\theta \in \Theta} b_\theta P_\theta} \leq \frac{1}{1 - \max_{\theta \in \Theta} \frac{\phi_\eta(\underline{l}_\theta)}{\phi_\eta(0)}} = \frac{1}{1 - \frac{\phi_\eta(\underline{l})}{\phi_\eta(0)}}.$$

By using the upper bound (17) for ρ_θ, we obtain

$$\begin{aligned}
&\frac{1 - \lambda + \max_{\theta \in \Theta} \kappa_\theta}{1 - \max_{\theta \in \Theta} b_\theta P_\theta} \\
\leq &\frac{1}{1 - \frac{\phi_\eta(\underline{l})}{\phi_\eta(0)}} \left[\frac{1}{\phi_\eta(0) - \phi_\eta(\underline{l})} + \frac{1}{\phi_\eta(0)} \right] \frac{3L}{2} \max_{\theta \in \Theta} \frac{\rho_\theta}{P_\theta} \\
\leq &\frac{1}{1 - \frac{\phi_\eta(\underline{l})}{\phi_\eta(0)}} \left[\frac{1}{\phi_\eta(0) - \phi_\eta(\underline{l})} + \frac{1}{\phi_\eta(0)} \right] \frac{3L}{2} \frac{2c}{[\phi_\eta(0) - \phi_\eta(\underline{l})] \underline{P}^2 B} \\
\leq &\frac{6 \phi_\eta(0) L c}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^3 \underline{P}^2 B}.
\end{aligned}$$

By adopting Lemma 2, we obtain that

$$\begin{aligned}
\sum_{\theta \in \Theta} z_\theta - \mathrm{LB} &\leq \left[1 - \frac{6 \phi_\eta(0) L c}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^3 \underline{P}^2 B} \right]^{-1} \mathrm{LB} - \mathrm{LB} \\
&\leq \frac{12 \phi_\eta(0) L c}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^3 \underline{P}^2 B} \mathrm{LB} \\
&\leq \frac{12 \phi_\eta(0)^2 L c^2}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^4 \underline{P}^2 B}.
\end{aligned}$$

As the last step of this proof, we derive the upper bound of $u_P^* - u_P(p)$ taking the baseline utility $\underline{u}_P$ into consideration, where u_P^* is defined to be the optimal principal utility any truth-telling incentivizing contract may achieve.

(1) $\sum_{\theta\in\Theta} z_\theta \le \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \underline{u}_P$. In this case, Algorithm 2 outputs the non-trivial contract discussed above. Since the contract simultaneously incentivizes truth-telling and positive agent effort, we have that

$$\begin{aligned} u_P^* - u_P(p) &\le \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \mathrm{LB} - \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) + \sum_{\theta\in\Theta} z_\theta \\ &\le \frac{12\phi_\eta(0)^2 L c^2}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^4 \underline{P}^2 B}. \end{aligned}$$

(2) $\sum_{\theta\in\Theta} z_\theta > \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \underline{u}_P$. In this case, Algorithm 2 outputs the zero-payment contract: $p(\hat{\theta}, x) = 0, \forall\hat{\theta}\in\Theta, x\in\mathbb{R}$. Similarly,

$$\begin{aligned} u_P^* - u_P(p) &\le \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \mathrm{LB} - \underline{u}_P \\ &\le \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \mathrm{LB} - \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) + \sum_{\theta\in\Theta} z_\theta \\ &\le \frac{12\phi_\eta(0)^2 L c^2}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^4 \underline{P}^2 B}. \end{aligned}$$

As a conclusion,

$$u_P^* - u_P(p) \le \frac{12\phi_\eta(0)^2 L c^2}{[\phi_\eta(0) - \phi_\eta(\underline{l})]^4 \underline{P}^2 B}.$$

Thus, we finish the proof of Theorem 5. □

In the main text, our theoretical analysis for Algorithm 2 (Theorem 5) does not hold when the noise distribution F_η is a zero-mean Gaussian distribution since it violates Assumption 3. Here we redo the analysis for Algorithm 2 for another family of noise distributions, containing Gaussian distributions.

Assumption 5. *The noise distribution F_η has a probability density function ϕ_η satisfying:*

- *ϕ_η is symmetric: $\phi_\eta(x) = \phi_\eta(-x), \forall x\in\mathbb{R}$,*
- *ϕ_η is monotonically non-increasing in $\mathbb{R}^+$: $\phi_\eta(x_1) \ge \phi_\eta(x_2), \forall 0\le x_1\le x_2$,*
- *$\phi_\eta(x-d)/\phi_\eta(x) \le \phi_\eta(-d)/\phi_\eta(0), \forall x\le 0, d\ge 0$.*
- *There exists a negative infinite sequence $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty}\phi_\eta(x_n - d)/\phi_\eta(x_n) = 0$ for any $d > 0$.*

Theorem 7. *Define $\bar{P} = \max_{\theta\in\Theta} P_\theta$. Let p be a contract generated by Algorithm 2 with inputs*

$$\rho_\theta = F_\eta^{-1}\left[\frac{1}{2} + \frac{c}{(1-\bar{P})BP_\theta}\right], \quad \forall\theta\in\Theta.$$

Suppose $\phi_\eta(\frac{1}{3}\underline{l}) \le \frac{(1-\bar{P})\underline{P}}{2(2-\underline{P})}\phi_\eta(0)$, $v(\theta,\theta)\ne v(\theta,\theta'), \forall\theta,\theta'\in\Theta$, and Assumption 2, 5 hold. Then $\exists B_0\in\mathbb{R}^+, \forall B\ge B_0$, contract p has the following properties:

(1) The budget constraint is never violated: $B_\theta \le B, \forall\theta\in\Theta$.
(2) The agent is incentivized to report the true state that he observes after the exploration.
(3) The difference between the principal utility generated by any truth-telling incentivizing contract p_0 and the utility induced by contract p is upper bounded as

$$u_P(p_0) - u_P(p) \le \frac{2(2-\underline{P})}{(1-\bar{P})\underline{P}}\,\frac{cm}{m-1}\,\frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)}.$$

Proof of Theorem 7. The selected payment radius for any $\theta\in\Theta$ is given by

$$\rho_\theta = F_\eta^{-1}\left[\frac{1}{2} + \frac{c}{(1-\bar{P})BP_\theta}\right].$$

We note that $\rho_\theta \to 0$ as $B\to+\infty$. Let p denote the output of Algorithm 2. If $\phi_\eta(\frac{1}{3}\underline{l}) \le \frac{(1-\bar{P})\underline{P}}{2(2-\underline{P})}\phi_\eta(0) \le \frac{1}{2}\phi_\eta(0)$, the demonstration that when $\sum_{\theta\in\Theta} z_\theta \le \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \underline{u}_P$, the contract p is ex-post bounded by B, incentivizes both positive agent effort and honest reporting is similar to that in the proof of Theorem 5. In this proof, we focus on demonstrating the efficiency of p in generating the principal's utility.

The key to our proof is to adopt Lemma 2, so we need to first compute the values of $a_{\theta,\theta'}, b_{\theta'}, a'_{\theta,\theta'}, b'_{\theta'}$ for any $\theta, \theta' \in \Theta : \theta \neq \theta'$. By Assumption 5, there exists an infinite sequence $\{s_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \frac{P_\theta \phi_\eta(s_n - v(\theta',\theta))}{P_{\theta'}\phi_\eta(s_n - v(\theta',\theta'))} = 0$ for any $\theta, \theta' \in \Theta : \theta \neq \theta'$. Thus, we have that $a_{\theta,\theta'} = 0$ and $b_{\theta'} = 1$ for any $\theta, \theta' \in \Theta$. In summary,

$$\begin{cases} a_{\theta,\theta'} &= 0, \\ a'_{\theta,\theta'} &= \dfrac{P_\theta \int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta}, \\ b_{\theta'} &= 1, \\ b'_{\theta'} &= \sum_{\theta\in\Theta} \dfrac{P_\theta \int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta}. \end{cases}$$

Now, we compute valid upper bounds $\epsilon_{\theta,\theta'}, \kappa_{\theta'}$ for $a'_{\theta,\theta'} - a_{\theta,\theta'}$ and $b'_{\theta'} - b_{\theta'}$, respectively. With $l_{\theta,\theta'} := v(\theta',\theta') - v(\theta',\theta)$ and $\underline{l} := \min_{\theta\neq\theta'} |l_{\theta,\theta'}|$, $\epsilon_{\theta,\theta'}$ is given by the following derivation

$$\begin{aligned} a'_{\theta,\theta'} - a_{\theta,\theta'} =& \frac{P_\theta \int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} = \frac{P_\theta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} \phi_\eta(x + l_{\theta,\theta'}) dx \\ =& \frac{P_\theta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} \phi_\eta(x - |l_{\theta,\theta'}|) dx \leq \frac{P_\theta \int_{-\rho_{\theta'}}^{\rho_{\theta'}} \phi_\eta(x) dx}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} \frac{\phi_\eta(\rho_{\theta'} - |l_{\theta,\theta'}|)}{\phi_\eta(\rho_{\theta'})} \\ \leq& \frac{P_\theta \phi_\eta(\frac{2}{3}\underline{l})}{P_{\theta'}\phi_\eta(\frac{1}{3}\underline{l})} \leq \frac{P_\theta \phi_\eta(\frac{1}{3}\underline{l})}{P_{\theta'}\phi_\eta(0)} \end{aligned}$$

when B is sufficiently large such that $\rho_{\theta'} \leq \frac{1}{3}\underline{l}$. $\kappa_{\theta'}$ is given by the following derivation

$$\begin{aligned} b'_{\theta'} - b_{\theta'} =& \sum_{\theta\in\Theta} \frac{P_\theta \int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} dF_\eta}{P_{\theta'} \int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta} - 1 \\ \leq& 1 + \sum_{\theta\in\Theta:\theta\neq\theta'} \frac{P_\theta \phi_\eta(\frac{1}{3}\underline{l})}{P_{\theta'}\phi_\eta(0)} - 1 = \frac{1 - P_{\theta'}}{P_{\theta'}} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)}. \end{aligned}$$

In summary, we set

$$\epsilon_{\theta,\theta'} = \frac{P_\theta \phi_\eta(\frac{1}{3}\underline{l})}{P_{\theta'}\phi_\eta(0)}, \quad \kappa_{\theta'} = \frac{1 - P_{\theta'}}{P_{\theta'}} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)}, \quad \forall \theta, \theta' \in \Theta.$$

The last task before we adopt Lemma 2 is to derive an upper bound for $1 - \lambda$:

$$\begin{aligned} 1 - \lambda \leq& \max_{\theta\neq\theta'} \frac{\epsilon_{\theta,\theta'}}{P_\theta - \epsilon_{\theta,\theta'}P_{\theta'} + \epsilon_{\theta,\theta'}} = \max_{\theta\neq\theta'} \frac{\epsilon_{\theta,\theta'}}{P_\theta + (1 - P_{\theta'})\epsilon_{\theta,\theta'}} \\ \leq& \max_{\theta\neq\theta'} \frac{\epsilon_{\theta,\theta'}}{P_\theta} = \frac{1}{\underline{P}} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)}. \end{aligned}$$

Say $\vec{z}$ is the solution to the program in Algorithm 2. Then the expected payment of contract p is $\sum_{\theta\in\Theta} z_\theta$. By Lemma 2, we have that

$$\begin{aligned} \sum_{\theta\in\Theta} z_\theta - \text{LB} \leq& \left\{ \left[1 - \frac{\frac{1}{\underline{P}} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)} + \max_{\theta\in\Theta} \frac{1-P_\theta}{P_\theta} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)}}{1 - \bar{P}} \right]^{-1} - 1 \right\} \cdot \text{LB} \\ =& \left\{ \left[1 - \frac{2 - \underline{P}}{(1 - \bar{P})\underline{P}} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)} \right]^{-1} - 1 \right\} \cdot \text{LB} \\ \leq& \frac{2(2 - \underline{P})}{(1 - \bar{P})\underline{P}} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)} \cdot \text{LB} \\ =& \frac{2(2 - \underline{P})}{(1 - \bar{P})\underline{P}} \frac{\phi_\eta(\frac{1}{3}\underline{l})}{\phi_\eta(0)} \frac{cm}{m - 1}, \end{aligned}$$

where the last equality is by Lemma 3 and the fact that $a_{\theta,\theta'} = 0, b_{\theta'} = 1$ for any $\theta, \theta' \in \Theta : \theta \neq \theta'$. □

Corollary 1. *Suppose $F_\eta = Gauss(0, \sigma^2)$ in Theorem 7. Then $\exists B_0 \in \mathbb{R}^+, \forall B \geq B_0$, the difference between the principal utility generated by any truth-telling incentivizing contract p_0 and the utility induced by contract p is upper bounded as*

$$u_P(p_0) - u_P(p) \leq \frac{2(2-\underline{P})}{(1-\bar{P})\underline{P}} \frac{cm}{m-1} \exp\left(-\frac{\underline{l}^2}{18\sigma^2}\right).$$

Here, we present Lemma 3, which was used in the proof of Theorem 7.

Lemma 3. *$\{w_i\}_{i=1}^m$ is a sequence of real numbers. If we have that $\sum_{i=1}^m w_i - u \geq \max_{i\in[m]} \lambda_i w_i$ for constants u and $\lambda_i \geq 1, \forall i \in [m]$ such that $\sum_{i=1}^m \frac{1}{\lambda_i} > 1$, then there exists a lower bound for $\sum_{i=1}^m w_i$:*

$$\sum_{i=1}^m w_i \geq u \cdot \frac{\sum_{i=1}^m \frac{1}{\lambda_i}}{\sum_{i=1}^m \frac{1}{\lambda_i} - 1}.$$

Moreover, this lower bound can be achieved if

$$w_i = u \cdot \frac{\frac{1}{\lambda_i}}{\sum_{j=1}^m \frac{1}{\lambda_j} - 1}, \quad \forall i \in [m]. \tag{18}$$

Proof of Lemma 3. Let k denote the index such that $\lambda_k w_k = \max_{i\in[m]} \lambda_i w_i$. That is, $\lambda_k w_k \geq \lambda_i w_i, \forall i \in [m]$. On the one hand,

$$\sum_{i=1}^m w_i = \sum_{i=1}^m \frac{\lambda_i w_i}{\lambda_i} \leq \sum_{i=1}^m \frac{\lambda_k w_k}{\lambda_i} = \lambda_k w_k \sum_{i=1}^m \frac{1}{\lambda_i}.$$

On the other hand, by our assumption $\sum_{i=1}^m w_i - u \geq \max_{i\in[m]} \lambda_i w_i$,

$$\sum_{i=1}^m w_i \geq u + \lambda_k w_k.$$

Merging the above two inequalities, we can derive a lower bound for $\lambda_k w_k$,

$$\lambda_k w_k \geq \frac{u}{\sum_{i=1}^m \frac{1}{\lambda_i} - 1}.$$

We use the assumption again to obtain

$$\sum_{i=1}^m w_i \geq u + \lambda_k w_k \geq u + \frac{u}{\sum_{i=1}^m \frac{1}{\lambda_i} - 1} = u \cdot \frac{\sum_{i=1}^m \frac{1}{\lambda_i}}{\sum_{i=1}^m \frac{1}{\lambda_i} - 1}.$$

Finally, we notice that when

$$w_i = u \cdot \frac{\frac{1}{\lambda_i}}{\sum_{j=1}^m \frac{1}{\lambda_j} - 1}, \quad \forall i \in [m],$$

we have $\lambda_i w_i = \max_{j\in[m]} \lambda_j w_j = u \cdot \frac{1}{\sum_{j=1}^m \frac{1}{\lambda_j} - 1}, \forall i \in [m]$. Furthermore,

$$\begin{aligned}\sum_{i=1}^m w_i - u = &\sum_{i=1}^m u \cdot \frac{\frac{1}{\lambda_i}}{\sum_{j=1}^m \frac{1}{\lambda_j} - 1} - u \\ =&u \cdot \frac{1}{\sum_{j=1}^m \frac{1}{\lambda_j} - 1} = \max_{j\in[m]} \lambda_j w_j,\end{aligned}$$

thus the constraint is not violated. Besides, it is straightforward to verify that the lower bound of $\sum_{i=1}^m w_i$ is achieved if $\{w_i\}_{i=1}^m$ is given by (18). ◻

C MISSING PROOFS FOR SECTION 4.3

In Section 4.3, we introduced a counterexample showing that the optimal contract does not necessarily incentivize truth-telling. Here, we formally present this result in Theorem 8.

Theorem 8 (Optimal Contract May Not Incentivize Truth-Telling). *There exists a problem instance such that the optimal contract does not incentivize truth-telling.*

Proof of Theorem 8. Let $m = 4, c = 1$, and $P_\theta = \frac{1}{m}, \forall\theta \in \Theta$. We consider the following principal value matrix:

$$v = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$$

The noise distribution F_η is set to be a Laplace distribution with a sufficiently large λ. Let θ_i denote the true state corresponding to the i-column of the value matrix. Let p^* denote the optimal truth-telling-incentivizing contract. By adopting Proposition 3 in the main text, an upper bound for the principal utility generated by p^* can be computed:

$$u_P(p^*) \le \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - \text{LB} = \frac{1}{4} \times 202 - (\frac{1}{6} + \frac{1}{6} + \frac{2}{3} + \frac{2}{3}) = 48 + \frac{5}{6}.$$

Consider a contract p satisfying the following requirements:

$$\int_{\mathbb{R}} p(\theta_1, x) dF_{X|\theta_1,\theta} = \left(\frac{1}{6} - \frac{13}{500}\right) \cdot m, \forall\theta \in \Theta,$$
$$\int_{\mathbb{R}} p(\theta_2, x) dF_{X|\theta_2,\theta} = \left(\frac{1}{6} + \frac{3}{50}\right) \cdot m, \forall\theta \in \{\theta_1, \theta_2\}, \quad \int_{\mathbb{R}} p(\theta_2, x) dF_{X|\theta_2,\theta} = 0, \forall\theta \in \{\theta_3, \theta_4\},$$
$$\int_{\mathbb{R}} p(\theta, x) dF_{X|\theta,\theta} = \left(\frac{2}{3} - \frac{1}{10}\right) \cdot m, \quad \int_{\mathbb{R}} p(\theta, x) dF_{X|\theta,\theta'} = 0, \forall\theta \in \{\theta_3, \theta_4\}, \theta' \in \Theta \; s.t. \; \theta' \ne \theta.$$

These requirements are reasonable since the construction of v ensures that $F_{X|\theta_1,\theta} = F_{X|\theta_1,\theta'}, \forall\theta,\theta' \in \Theta$, and $F_{X|\theta_2,\theta_1} = F_{X|\theta_2,\theta_2}$. We observe that contract p does **NOT** incentivize truth-telling: when he explores and observes a true state θ_1,

$$\int_{\mathbb{R}} p(\theta_2, x) dF_{X|\theta_2,\theta_1} = \left(\frac{1}{6} + \frac{3}{50}\right) \cdot m > \int_{\mathbb{R}} p(\theta_1, x) dF_{X|\theta_1,\theta_1} = \left(\frac{1}{6} - \frac{13}{500}\right) \cdot m,$$

so he will choose to report θ_2 instead of the true state θ_1, i.e., $\bar{r}(\theta_1) = \theta_2$. It can be verified that $\bar{r}(\theta) = \theta, \forall\theta \ne \theta_1$. The agent's utility when he chooses zero effort is

$$\max_{\theta'\in\Theta} \sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta} = \frac{2}{3} - \frac{1}{10}.$$

The agent's utility when he chooses positive effort is

$$\begin{aligned} &\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p\big(\bar{r}(\theta), x\big) dF_{X|\bar{r}(\theta),\theta} - c \\ =&2\left(\frac{1}{6} + \frac{3}{50}\right) + 2\left(\frac{2}{3} - \frac{1}{10}\right) - 1 = \frac{2}{3} - \frac{2}{25} \\ >&\frac{2}{3} - \frac{1}{10} = \max_{\theta'\in\Theta} \sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta', x) dF_{X|\theta',\theta}. \end{aligned}$$

Thus, contract p incentivizes the agent to take positive effort. We can now compute the principal utilited generated by contract p:

$$\begin{aligned} u_P(p) &= \sum_{\theta\in\Theta} P_\theta v\big(\bar{r}(\theta), \theta\big) - \sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p\big(\bar{r}(\theta), x\big) dF_{X|\bar{r}(\theta),\theta} \\ &= \frac{1}{4} \times 202 - \left[2\left(\frac{1}{6} + \frac{3}{50}\right) + 2\left(\frac{2}{3} - \frac{1}{10}\right)\right] = 48 + \frac{5}{6} + \frac{2}{25} > 48 + \frac{5}{6} \ge u_P(p^*). \end{aligned}$$

As a summary, we have shown that the principal utility generated by the optimal truth-telling-incentivizing contract p^* is dominated by that of a contract p that does not incentivize truth-telling. The proof is complete. □

We now introduce a special type of contract such that for each report θ, its payment is centered at around $v(\theta,\theta)$ and the value of the payment is proportional to the value of $v(\theta,\theta)$. That is, $\exists k > 0$ such that

$$\int_{\mathbb{R}} p(\theta, x) dF_{X|\theta,\theta} = kv(\theta,\theta), \quad \forall\theta \in \Theta.$$

Definition 9 (Diagonal Proportional Bounded Dirac Delta Contract (`Diag-BDD`)). For any agent report $\hat{\theta} \in \Theta$, the `Diag-BDD`(B, k) contract is given by

$$p(\hat{\theta}, x) = \begin{cases} B, & \text{if } |x - v(\hat{\theta}, \hat{\theta})| \le \rho_{\hat{\theta}}, \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

where $\rho_{\hat{\theta}} := F_\eta^{-1}\left[\frac{1}{2} + \frac{kv(\hat{\theta},\hat{\theta})}{2B}\right]$.

We show in the following lemma that when Assumption 2 holds with a sufficiently large δ, the `Diag-BDD` contract incentivizes truth-telling under positive agent effort. ϕ_η^{-1} is the inverse function of ϕ_η. Lemma 4 will be used in the proof of Theorem 6.

Lemma 4. *Consider a* `Diag-BDD` *contract* p. *For any* $\theta \in \Theta$, $\bar{\rho}_\theta \geq 0$ *is a known upper bound for* ρ_θ. *Suppose that Assumption 4 holds and at least one of the following conditions holds:*

(1) Assumption 2 holds with $\delta \geq$

$$\max_{\theta,\theta'\in\Theta} \min\left\{y \geq 2\bar{\rho}_{\theta'} \Big| \frac{\phi_\eta(x+y-\bar{\rho}_{\theta'})}{\phi_\eta(\bar{\rho}_{\theta'})} \leq \frac{v(\theta,\theta)}{x+v(\theta,\theta)}, \forall x \geq 0\right\}.$$

(2) An upper bound $\bar{v}$ *is known for* $\max_{\theta,\theta'\in\Theta} v(\theta,\theta')$ *and Assumption 2 holds with*

$$\delta \geq \max_{\theta,\theta'\in\Theta} \bar{\rho}_{\theta'} + \phi_\eta^{-1}\Big[\frac{v(\theta,\theta)}{\bar{v}}\phi_\eta(\bar{\rho}_{\theta'})\Big].$$

Then contract p *incentivizes truth-telling under positive agent effort.*

Proof of Lemma 4. Consider a `Diag-BDD` contract p characterized by B, k. We attempt to show, for any $\theta, \theta' \in \Theta$, the contract p satisfies that

$$\int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} \geq \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta}, \tag{20}$$

which means that reporting a false value θ' never results in a higher expected payment than reporting the truth θ.

There are two possible cases:

(a) $v(\theta,\theta) \geq v(\theta',\theta')$. We first show that the following inequality holds:

$$\int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta} \leq \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta'}. \tag{21}$$

Define $l = v(\theta',\theta') - v(\theta',\theta)$. We only consider the case where $l \geq 0$, as the proof is symmetric otherwise. By definition of the `Diag-BDD` contract, we have

$$\int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta'} = \int_{v(\theta',\theta')-\rho_{\theta'}}^{v(\theta',\theta')+\rho_{\theta'}} BdF_{X|\theta',\theta'} = \int_{-\rho_{\theta'}}^{\rho_{\theta'}} BdF_\eta$$

and

$$\int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta} = \int_{v(\theta',\theta')-\rho_{\theta'}}^{v(\theta',\theta')+\rho_{\theta'}} BdF_{X|\theta',\theta} = \int_{l-\rho_{\theta'}}^{l+\rho_{\theta'}} BdF_\eta.$$

Due to the monotonicity and symmetry of ϕ_η (Assumption 4), when $l \geq 2\rho_{\theta'}$,

$$\begin{aligned}
&\int_{-\rho_{\theta'}}^{\rho_{\theta'}} BdF_\eta - \int_{l-\rho_{\theta'}}^{l+\rho_{\theta'}} BdF_\eta \\
\geq &\int_{-\rho_{\theta'}}^{\rho_{\theta'}} B\phi_\eta(\rho_{\theta'})dx - \int_{l-\rho_{\theta'}}^{l+\rho_{\theta'}} B\phi_\eta(l-\rho_{\theta'})dx \\
\geq &\int_{-\rho_{\theta'}}^{\rho_{\theta'}} B\phi_\eta(\rho_{\theta'})dx - \int_{l-\rho_{\theta'}}^{l+\rho_{\theta'}} B\phi_\eta(\rho_{\theta'})dx = 0.
\end{aligned}$$

Otherwise, when $0 \leq l < 2\rho_{\theta'}$, we similarly have

$$\begin{aligned}
&\int_{-\rho_{\theta'}}^{\rho_{\theta'}} BdF_\eta - \int_{l-\rho_{\theta'}}^{l+\rho_{\theta'}} BdF_\eta \\
= &\int_{-\rho_{\theta'}}^{l-\rho_{\theta'}} B\big[\phi_\eta(x) - \phi_\eta(x+2\rho_{\theta'})\big]dx \\
\geq &\int_{-\rho_{\theta'}}^{l-\rho_{\theta'}} B\big[\phi_\eta(-\rho_{\theta'}) - \phi_\eta(\rho_{\theta'})\big]dx = 0.
\end{aligned}$$

Thus, (21) is proved. Our claim (21) directly implies (20):

$$\begin{aligned}
&\int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} = kv(\theta,\theta) \geq kv(\theta',\theta') = \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta'} \\
\geq &\int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta}.
\end{aligned}$$

(b) $v(\theta,\theta) < v(\theta',\theta')$. We first show that the following inequality holds under different assumptions:

$$\frac{\phi_\eta\big(v(\theta',\theta') - v(\theta,\theta) + \delta - \bar{\rho}_{\theta'}\big)}{\phi_\eta(\bar{\rho}_{\theta'})} v(\theta',\theta') \le v(\theta,\theta). \tag{22}$$

If we have that δ is not smaller than

$$\underline{\delta}_{\theta,\theta'} := \min\Big\{y \ge 2\bar{\rho}_{\theta'} \Big| \frac{\phi_\eta(x + y - \bar{\rho}_{\theta'})}{\phi_\eta(\bar{\rho}_{\theta'})} \le \frac{v(\theta,\theta)}{x + v(\theta,\theta)}, \forall x \ge 0\Big\},$$

then by the monotonicity of ϕ_η,

$$\begin{aligned}&\phi_\eta\big(v(\theta',\theta') - v(\theta,\theta) + \delta - \bar{\rho}_{\theta'}\big)\\ \le&\phi_\eta\big(v(\theta',\theta') - v(\theta,\theta) + \underline{\delta}_{\theta,\theta'} - \bar{\rho}_{\theta'}\big)\end{aligned}$$

since $v(\theta',\theta') > v(\theta,\theta)$ and $\delta \ge \underline{\delta}_{\theta,\theta'} \ge 2\bar{\rho}_{\theta'}$. Then we observe that (22) holds by the definition of $\underline{\delta}_{\theta,\theta'}$.

If we know $\bar{v} = \max_{\theta,\theta'\in\Theta} v(\theta,\theta')$ and that

$$\delta \ge \bar{\rho}_{\theta'} + \phi_\eta^{-1}\Big[\frac{v(\theta,\theta)}{\bar{v}}\phi_\eta(\bar{\rho}_{\theta'})\Big],$$

also by the monotonicity of ϕ_η,

$$\begin{aligned}&\phi_\eta\big(v(\theta',\theta') - v(\theta,\theta) + \delta - \bar{\rho}_{\theta'}\big)\\ \le&\phi_\eta\big(\delta - \bar{\rho}_{\theta'}\big) \le \frac{v(\theta,\theta)}{\bar{v}}\phi_\eta(\bar{\rho}_{\theta'}) \le \frac{v(\theta,\theta)}{v(\theta',\theta')}\phi_\eta(\bar{\rho}_{\theta'}).\end{aligned}$$

Thus, we have proved the correctness of (22). Now,

$$\begin{aligned}&\int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} = kv(\theta,\theta)\\ \ge&\frac{\phi_\eta\big(v(\theta',\theta') - v(\theta,\theta) + \delta - \bar{\rho}_{\theta'}\big)}{\phi_\eta(\bar{\rho}_{\theta'})}kv(\theta',\theta')\\ \ge&\frac{\phi_\eta\big(v(\theta',\theta') - v(\theta,\theta) + \delta - \rho_{\theta'}\big)}{\phi_\eta(\rho_{\theta'})}kv(\theta',\theta')\\ \ge&\frac{\phi_\eta(l - \rho_{\theta'})}{\phi_\eta(\rho_{\theta'})}\int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta'}\\ \ge&\int_{l-\rho_{\theta'}}^{l+\rho_{\theta'}} BdF_\eta = \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta}.\end{aligned}$$

In the above derivation, $l = v(\theta',\theta') - v(\theta',\theta)$, the second inequality is by recalling that $\rho_{\theta'} \le \bar{\rho}_{\theta'}$, while the final inequality is because the monotonicity of ϕ_η implies

$$\frac{\int_{-\rho_{\theta'}}^{\rho_{\theta'}} dF_\eta}{2\rho_{\theta'}\phi_\eta(\rho_{\theta'})} \ge 1 \ge \frac{\int_{l-\rho_{\theta'}}^{l+\rho_{\theta'}} dF_\eta}{2\rho_{\theta'}\phi_\eta(l - \rho_{\theta'})}$$

when $l \ge v(\theta',\theta') - v(\theta,\theta) + \delta > \delta \ge \bar{\rho}_{\theta'}$. □

Proof of Theorem 6. In this proof, for any contract p_0 such that $\exists\theta\in\Theta, \bar{r}(\theta) \ne \theta$, we construct a `Diag-BDD` contract p that satisfies all three properties mentioned in Theorem 6.

First, we introduce some definitions and assumptions. For any fixed parameter $\lambda, \nu \in (0,1)$ such that $\lambda > \nu$, we define

$$\bar{\rho}_\theta = F_\eta^{-1}\Bigg[\frac{1}{2} + \frac{v(\theta,\theta)c}{B\sum_{\tilde{\theta}\in\Theta} P_{\tilde{\theta}} v(\tilde{\theta},\tilde{\theta})}\Big(1 + \frac{2}{\lambda}\Big)\Bigg], \quad \forall\theta\in\Theta.$$

Note that $\bar{\rho}_\theta \to 0$ as $B \to +\infty$. We construct a `Diag-BDD` contract p whose parameter k is defined by the following equation,

$$\sum_{\theta\in\Theta} P_\theta kv(\theta,\theta) - c = \max_{\theta'\in\Theta} P_{\theta'}kv(\theta',\theta')\frac{\sum_{\theta\in\Theta} P_\theta\phi_\eta\big(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\big)}{P_{\theta'}\phi_\eta(\bar{\rho}_{\theta'})}.$$

Recall that $\bar{v} = \max_{\theta\in\Theta} v(\theta,\theta)$ and $\underline{v} = \min_{\theta\in\Theta} v(\theta,\theta)$. We assume δ, B to be sufficiently large. Specifically,

$$\delta \geq \lambda\bar{v} \vee \frac{1}{2\underline{P}}\left[c + \sqrt{c^2 + 16c\bar{v}\underline{P}}\right], \tag{23}$$

$$\bar{\rho}_\theta \leq \nu\bar{v}, \quad \forall\theta \in \Theta, \tag{24}$$

$$\phi_\eta(\bar{\rho}_\theta) \geq \frac{8}{8+\lambda}\phi_\eta(0), \quad \forall\theta \in \Theta, \tag{25}$$

$$\phi_\eta(\delta - \nu\bar{v}) \leq \frac{8}{8+\lambda}\frac{\underline{v}}{\bar{v}}\phi_\eta(0), \tag{26}$$

$$\phi_\eta(\delta/2 - \nu\bar{v}) \leq \frac{\lambda}{8+\lambda}\underline{P}\phi_\eta(0). \tag{27}$$

Step 1. Truthfulness.

We show that contract p incentivizes truth-telling. For any $\theta, \theta' \in \Theta$,

$$\phi_\eta(\delta - \bar{\rho}_{\theta'}) \leq \phi_\eta(\delta - \nu\bar{v}) \leq \frac{8}{8+\lambda}\frac{\underline{v}}{\bar{v}}\phi_\eta(0) \leq \frac{\underline{v}}{\bar{v}}\phi_\eta(\bar{\rho}_{\theta'}) \leq \frac{v(\theta,\theta)}{\bar{v}}\phi_\eta(\bar{\rho}_{\theta'})$$

by (24), (25), and (26). Thus, Lemma 4 implies that p incentivizes truth-telling.

Step 2. Derive an upper bound for the value of k and the expected payment of p.

There are two cases:

(1) $\sum_{\theta: v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}} P_\theta \geq \underline{P} > 0$. We first attempt to bound the following term with any $\theta' \in \Theta$,

$$\begin{aligned}
&v(\theta',\theta')\frac{\sum_{\theta\in\Theta} P_\theta\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})} \\
=&\frac{v(\theta',\theta')}{\phi_\eta(\bar{\rho}_{\theta'})}\Bigg[\sum_{\theta: v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}} P_\theta\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right) \\
&+ \sum_{\theta: v(\theta,\theta)-v(\theta',\theta')\leq\frac{\delta}{2}} P_\theta\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right)\Bigg] \\
\leq&\frac{v(\theta',\theta')}{\phi_\eta(\bar{\rho}_{\theta'})}\Bigg[\phi_\eta(0)\sum_{\theta: v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}} P_\theta \\
&+ \phi_\eta\left(\frac{\delta}{2} - \bar{\rho}_{\theta'}\right)\sum_{\theta: v(\theta,\theta)-v(\theta',\theta')\leq\frac{\delta}{2}} P_\theta\Bigg] \\
\leq&\epsilon + \frac{v(\theta',\theta')}{\phi_\eta(\bar{\rho}_{\theta'})}\phi_\eta(0)\sum_{\theta: v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}} P_\theta.
\end{aligned}$$

In the first inequality of the derivation above, we used the fact that $v(\theta,\theta) - v(\theta',\theta') \leq \frac{\delta}{2}$ implies

$$\begin{aligned}
&v(\theta',\theta') - v(\theta',\theta) \\
\geq&v(\theta,\theta) - \frac{\delta}{2} - v(\theta',\theta) \\
\geq&\delta - \frac{\delta}{2} = \frac{\delta}{2}.
\end{aligned}$$

We claim that the following inequality holds by choosing $\epsilon = \frac{\delta}{8}\sum_{\theta: v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}} P_\theta$:

$$\delta \geq 2\bar{\rho}_{\theta'} + 2\phi_\eta^{-1}\left[\frac{\epsilon}{v(\theta',\theta')}\phi_\eta(\bar{\rho}_{\theta'})\right], \quad \forall\theta' \in \Theta, \tag{28}$$

which explains the last inequality in the derivation above. Now, to prove this claim, we have

$$
\begin{aligned}
&\phi_\eta(\delta/2-\bar{\rho}_{\theta'})\\
\leq&\phi_\eta(\delta/2-\nu\bar{v}) \leq \frac{\lambda}{8+\lambda}\underline{P}\phi_\eta(0)\\
\leq&\frac{\bar{v}\lambda}{v(\theta',\theta')(8+\lambda)}\underline{P}\phi_\eta(0) \leq \frac{\delta}{v(\theta',\theta')(8+\lambda)}\underline{P}\phi_\eta(0)\\
\leq&\frac{\delta}{8v(\theta',\theta')}\underline{P}\phi_\eta(\bar{\rho}_{\theta'}) \leq \frac{\phi_\eta(\bar{\rho}_{\theta'})}{v(\theta',\theta')}\epsilon
\end{aligned}
$$

by (23), (24), (25), and (27), which proves inequality (28).

We further have that

$$
\begin{aligned}
&v(\theta',\theta')\frac{\sum_{\theta\in\Theta}P_\theta\phi_\eta\left(|v(\theta',\theta')-v(\theta',\theta)|-\bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}\\
\leq&v(\theta',\theta')\left[\frac{\delta}{8v(\theta',\theta')}+\frac{\phi_\eta(0)}{\phi_\eta(\bar{\rho}_{\theta'})}\right]\sum_{\theta:v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}}P_\theta.
\end{aligned}
$$

From the above upper bound, we have that for θ',

$$
\begin{aligned}
&\frac{\sum_{\theta\in\Theta}P_\theta v(\theta,\theta)}{v(\theta',\theta')\frac{\sum_{\theta\in\Theta}P_\theta\phi_\eta\left(|v(\theta',\theta')-v(\theta',\theta)|-\bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}}\\
\geq&\frac{\sum_{\theta:v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}}P_\theta v(\theta,\theta)}{v(\theta',\theta')\left[\frac{\delta}{8v(\theta',\theta')}+\frac{\phi_\eta(0)}{\phi_\eta(\bar{\rho}_{\theta'})}\right]\sum_{\theta:v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}}P_\theta}\\
\geq&\frac{\sum_{\theta:v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}}P_\theta\left[v(\theta',\theta')+\frac{\delta}{2}\right]}{v(\theta',\theta')\left[\frac{\delta}{8v(\theta',\theta')}+\frac{\phi_\eta(0)}{\phi_\eta(\bar{\rho}_{\theta'})}\right]\sum_{\theta:v(\theta,\theta)-v(\theta',\theta')>\frac{\delta}{2}}P_\theta}\\
=&\frac{1+\frac{\delta}{2v(\theta',\theta')}}{\frac{\delta}{8v(\theta',\theta')}+\frac{\phi_\eta(0)}{\phi_\eta(\bar{\rho}_{\theta'})}}.
\end{aligned}
$$

As a consequence, the expected payment of this contract p is upper bounded as

$$
\begin{aligned}
&\sum_{\theta\in\Theta}P_\theta kv(\theta,\theta)\\
\leq&c\max_{\theta'\in\Theta}\frac{1+\frac{\delta}{2v(\theta',\theta')}}{1+\frac{\delta}{2v(\theta',\theta')}-\frac{\delta}{8v(\theta',\theta')}-\frac{\phi_\eta(0)}{\phi_\eta(\bar{\rho}_{\theta'})}}\\
\leq&c\max_{\theta'\in\Theta}\frac{1+\frac{\delta}{2v(\theta',\theta')}}{\frac{\delta}{4v(\theta',\theta')}}=2c\left(1+\frac{2\bar{v}}{\delta}\right),
\end{aligned}
$$

which used a fact that

$$
\frac{\phi_\eta(0)}{\phi_\eta(\bar{\rho}_{\theta'})}\leq 1+\frac{\lambda}{8}\leq 1+\frac{\delta}{8\bar{v}}\leq 1+\frac{\delta}{8v(\theta',\theta')}
$$

by (23) and (25).

(2) $\sum_{\theta: v(\theta,\theta)-v(\theta',\theta') > \frac{\delta}{2}} P_\theta = 0$. We again bound the following term with any $\theta' \in \Theta$,

$$\begin{aligned}
&v(\theta',\theta')\frac{\sum_{\theta\in\Theta} P_\theta \phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}\\
=&\frac{v(\theta',\theta')}{\phi_\eta(\bar{\rho}_{\theta'})}\left[\sum_{\theta: v(\theta,\theta)-v(\theta',\theta') \le \frac{\delta}{2}} P_\theta \phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right)\right]\\
\le&\frac{v(\theta',\theta')}{\phi_\eta(\bar{\rho}_{\theta'})}\left[\phi_\eta\left(\frac{\delta}{2} - \bar{\rho}_{\theta'}\right)\sum_{\theta: v(\theta,\theta)-v(\theta',\theta') \le \frac{\delta}{2}} P_\theta\right]\\
\le&\frac{\delta}{8}\underline{P}.
\end{aligned}$$

The last inequality holds since

$$\begin{aligned}
&\phi_\eta(\delta/2 - \bar{\rho}_{\theta'}) \le \phi_\eta(\delta/2 - v\bar{v}) \le \frac{\lambda}{8+\lambda}\underline{P}\phi_\eta(0)\\
\le&\frac{\bar{v}\lambda}{v(\theta',\theta')(8+\lambda)}\underline{P}\phi_\eta(0) \le \frac{\delta}{v(\theta',\theta')(8+\lambda)}\underline{P}\phi_\eta(0)\\
\le&\frac{\delta}{8v(\theta',\theta')}\underline{P}\phi_\eta(\bar{\rho}_{\theta'})
\end{aligned}$$

by (23), (24), (25), and (27).

The expected payment of this contract p is upper bounded as

$$\begin{aligned}
&\sum_{\theta\in\Theta} P_\theta k v(\theta,\theta)\\
\le& c\max_{\theta'\in\Theta}\frac{\sum_{\theta\in\Theta} P_\theta v(\theta,\theta)}{\sum_{\theta\in\Theta} P_\theta v(\theta,\theta) - v(\theta',\theta')\frac{\sum_{\theta\in\Theta} P_\theta\phi_\eta\left(|v(\theta',\theta')-v(\theta',\theta)|-\bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}}\\
\le& c\frac{1}{1-\frac{\delta\underline{P}/8}{\sum_{\theta\in\Theta} P_\theta v(\theta,\theta)}} \le c\frac{1}{1-\frac{\delta\underline{P}/8}{m\underline{P}\delta}} < 2c.
\end{aligned}$$

Merging our discussion of both cases, the expected payment's upper bound is

$$\sum_{\theta\in\Theta} P_\theta k v(\theta,\theta) \le 2c\left(1+\frac{2\bar{v}}{\delta}\right),$$

which also implies an upper bound for the value of k. We observe that for any $\theta \in \Theta$, $\bar{\rho}_\theta$ is a valid upper bound for ρ_θ:

$$\rho_\theta = F_\eta^{-1}\left[\frac{1}{2} + \frac{kv(\theta,\theta)}{2B}\right] \le F_\eta^{-1}\left[\frac{1}{2} + \frac{v(\theta,\theta)c}{B\sum_{\theta'\in\Theta} P_{\theta'} v(\theta',\theta')}\left(1+\frac{2\bar{v}}{\delta}\right)\right] \le \bar{\rho}_\theta.$$

Step 3. Incentivizes positive effort.

We verify that the `Diag-BDD` contract associated with this k incentivizes positive agent effort: for any $\theta' \in \Theta$,

$$\begin{aligned}
&\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta',x)dF_{X|\theta',\theta}\\
=&\sum_{\theta\in\Theta} P_\theta \int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} BdF_\eta\\
\le&\sum_{\theta\in\Theta} P_\theta \frac{\phi_\eta\left(|v(\theta',\theta')-v(\theta',\theta)|-\bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}\int_{-\rho_{\theta'}}^{\rho_{\theta'}} BdF_\eta\\
=&kv(\theta',\theta')\sum_{\theta\in\Theta} P_\theta \frac{\phi_\eta\left(|v(\theta',\theta')-v(\theta',\theta)|-\bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}\\
\le&\sum_{\theta\in\Theta} P_\theta k v(\theta,\theta) - c = \sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p(\theta,x)dF_{X|\theta,\theta} - c.
\end{aligned}$$

We elaborate more on the first inequality in the above derivation. For any $\theta \in \Theta$, consider two possible cases:

(a) $|v(\theta',\theta') - v(\theta',\theta)| \le 2\bar{\rho}_{\theta'}$. Recalling the fact we have shown in the proof of Lemma 4 that

$$\int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} BdF_\eta \le \int_{-\rho_{\theta'}}^{\rho_{\theta'}} BdF_\eta$$

and noticing

$$\frac{\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})} \ge 1,$$

we obtain

$$\begin{aligned}&\int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} BdF_\eta\\ \le&\frac{\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}\int_{-\rho_{\theta'}}^{\rho_{\theta'}} BdF_\eta.\end{aligned}$$

(b) $|v(\theta',\theta') - v(\theta',\theta)| > 2\bar{\rho}_{\theta'}$. This implies $|v(\theta',\theta') - v(\theta',\theta)| > 2\rho_{\theta'}$, under which by the monotonicity of ϕ_η we have

$$\begin{aligned}&\int_{v(\theta',\theta')-v(\theta',\theta)-\rho_{\theta'}}^{v(\theta',\theta')-v(\theta',\theta)+\rho_{\theta'}} BdF_\eta\\ \le&\frac{\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \rho_{\theta'}\right)}{\phi_\eta(\rho_{\theta'})}\int_{-\rho_{\theta'}}^{\rho_{\theta'}} BdF_\eta.\end{aligned}$$

The last inequality of case (a) is obtained from the fact that

$$\frac{\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \rho_{\theta'}\right)}{\phi_\eta(\rho_{\theta'})} \le \frac{\phi_\eta\left(|v(\theta',\theta') - v(\theta',\theta)| - \bar{\rho}_{\theta'}\right)}{\phi_\eta(\bar{\rho}_{\theta'})}.$$

Thus, we have validated that given this choice of k, the contract p incentivizes positive agent effort.

Step 4. Comparing p and p_0.

We turn to consider contract p_0. We will show that it does not yield a higher expected utility for the principal than p does when δ is sufficiently large. Let $\bar{r}(\theta)$ denote the report incentivized by p_0 when the state is θ. The expected principal utility generated by p_0 is

$$u_P(p_0) = \sum_{\theta\in\Theta} P_\theta v(\bar{r}(\theta),\theta) - \sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p_0(\bar{r}(\theta),x)dF_{X|\bar{r}(\theta),\theta},$$

subject to the positive agent effort incentive compatibility

$$\begin{aligned}&\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p_0(\bar{r}(\theta),x)dF_{X|\bar{r}(\theta),\theta} - c\\ \ge& \max_{\theta'\in\Theta}\sum_{\theta\in\Theta} P_\theta \int_{\mathbb{R}} p_0(\theta',x)dF_{X|\theta',\theta} \ge 0.\end{aligned}$$

As a final step, we compare the expected principal utility generated by contract p_0 and the `Diag-BDD` contract parameterized by B, k:

$$\begin{aligned}&u_P(p_0) - u_P(p)\\ \le& \sum_{\theta\in\Theta} P_\theta v(\bar{r}(\theta),\theta) - c - \sum_{\theta\in\Theta} P_\theta v(\theta,\theta) + 2c\left(1+\frac{2\bar{v}}{\delta}\right)\\ \le& -\sum_{\theta\in\Theta} P_\theta \delta \mathbb{I}\{\bar{r}(\theta)\ne\theta\} - c + 2c\left(1+\frac{2\bar{v}}{\delta}\right)\\ \le& -\underline{P}\delta - c + 2c\left(1+\frac{2\bar{v}}{\delta}\right) \le 0\end{aligned}$$

when

$$\delta \ge \frac{1}{2\underline{P}}\left[c + \sqrt{c^2 + 16c\bar{v}\underline{P}}\right]$$

holds due to (23). □