Submitted to Econometrica

Optimal Bayesian Commitments in Asymmetric Auctions with Incomplete Information

January 5, 2017

Submitted to Econometrica

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Abstract

We solve the Bayesian sequential equilibrium of a general class of single-item first-price or all-pay asymmetric auctions of incomplete information. Our main contribution is a general methodology for solving the optimal Bayesian commitment problem, in closed form, for asymmetric continuous-type distributions. The optimal commitment functions in these auctions reveal some important insights. When the player with commitment power (the leader) has low valuation, as a credible way to alleviate competition and to enable collusion, he bids passively. In a Bayesian commitment setting, the leader's type follows a distribution and may change in each period. Our approach consists of a number of methodological innovations. We propose a modeling concept called equal-bid function to simplify strategic interactions between the two players. We also introduce a concept called equal-utility curve to transform any commitment strategy into a weakly better continuous and differentiable strategy.

KEYWORDS: all-pay auction, asymmetric auction, Bayesian sequential equilibrium, commitment, first-price auction.

1. INTRODUCTION

First-price auctions have seen many applications because they are simple, intuitive and easy to implement. In first-price auctions, the highest bidder wins and pays her own bid. Theoretically, this auction format enjoys many desirable modeling properties. For example, in symmetric settings, this auction format has a unique and efficient Bayes Nash Equilibrium (BNE) (Chawla and Hartline 2013). In contrast, second-price auctions may have many inefficient equilibria.

At the same time, first-price auctions pose important challenges for both academics and practitioners. For one, in complete-information settings, the first-price auction format, including the class of *generalized first-price auctions*, sometimes does not have a pure Nash Equilibrium and is practically observed to be unstable (Edelman and Ostrovsky 2007; Edelman et al. 2007; Börgers et al. 2013). In incomplete-information settings where bidders have asymmetric type distributions, it is extremely difficult to solve or characterize its BNE. In fact, this has been one of the most elusive open problems in the literature of auction analysis (Vickrey 1961; Lebrun 1999; Fibich and Gavish 2011; Hartline et al. 2014). To date, the problem has closed-form solutions only in very restrictive settings such as two-bidder asymmetric uniform distributions (Kaplan and Zamir 2012; Fibich and Gavish 2012).

In this paper, we examine a general class of first-price auctions in which a single leader is capable of making commitments to his strategy before a follower subsequently chooses her strategy to maximize her payoff. Different from the setting of complete-information, symmetric first-price auction games, in this game the players have *asymmetric* type distributions and the players' types may be random draws from period to period.

Such a setting, known as Bayesian Stackelberg games (Fudenberg and Tirole 1991), is both theoretically interesting and practically meaningful. Theoretically, Fudenberg and Levine (1989) propose a Stackelberg reputation game in which a long-run player can enjoy a premium by making upfront commitments. Our result improves on theirs mainly in two ways. First, the leader in our setting will have random draws to determine his type in each period. The "Bayesian" in the name of the game refers to the fact that the strategy will be a mapping from type distributions to actions. Second, due to the nature of the auction setting, the information is incomplete and the players necessarily have to have asymmetric type distributions. These two improvements makes the model more general, but at the same time, it is known to be challenging to solve the game with these new features.

Our research is motivated by commitments in first-price auctions in practice. In flower auctions, some large players may have daily variations in their values, but they can make credible commitments by following a pre-announced strategy and making their past bids publicly verifiable. Such commitments can bring them extra payoffs compared to the case when daily auctions are treated independently. As an application in a different setting (Paruchuri et al. 2007; Tambe 2011),[1] the Los Angeles International Airport (LAX) adopted an algorithm called ARMOR (Assistant for Randomized Monitoring of Routes) to randomize the patrols. At the core of the algorithm is a game of a leader (the law enforcement) committing to a publicly known strategy that achieves the highest payoff against the follower (the terrorist).

In a Stackelberg equilibrium, given that it is possible for a follower to know a leader's committed strategy in advance and best-responds to it, the leader solves for an optimal strategy to commit to. A Stackelberg equilibrium formulation is particularly useful when one player has credibility to commit. It is well known that commitment weakly increases the leader's utility compared to his payoff in a simultaneous-move Nash equilibrium. Furthermore, there are efficient algorithms to compute it in basic settings (Conitzer and Sandholm 2006; Letchford and Conitzer 2010). It is not difficult to argue that the leader's type can have two components: a static component that does not change from period to period (for example, consumers may have a relatively constant preference for flowers; the cost of maintaining a patrol team at the airport may be quite predictable on a daily basis) and a dynamic part that may change over time (for example, weather may influence demand for flowers; terrorist activities may change the need for patrols). Given these factors that change from period to period, we study a Bayesian

[1]See also http://www.newsweek.com/random-security-laxs-armor-system-103885 for a *Newsweek* report.

Stackelberg game to accommodate such forms of uncertainty.

1.1. Commitment

We start our analysis by demonstrating that a leader's commitment in a sequential game yields higher profits compared to a simultaneous-move game.

Consider the following example in a first-price auction.[2] There are two bidders. One of the bidders, we call him leader A, has the power to commit to a bidding strategy first. Then the other bidder, we call her follower B, best-responds to the leader's commitment.

Example 1.1 *Suppose both players' types, x and y, are drawn uniformly from $[0,1]$. In a simultaneous-move game, the unique symmetric BNE suggests that each bidder bids half of the value, and player A's expected revenue is $1/6 \approx 0.167$. In contrast, in a sequential game, consider the leader's strategy $s_A(x) = x^2/2$. Clearly, the follower B must never bid more than 0.5 in this case. In fact, her utility when bidding $t \leq 0.5$ is $(y-t)\sqrt{2t}$. B's best strategy is $s_B(y) = y/3$. The expected utility of A is:*

$$\int_0^{\sqrt{2/3}} (x - \frac{1}{2}x^2) \cdot \frac{3}{2}x^2 dx + \int_{\sqrt{2/3}}^1 (x - \frac{1}{2}x^2) dx = 0.2029,$$

where the first term considers the case when x is in $[0, \sqrt{2/3}]$, and the second term considers the case when x is in $(\sqrt{2/3}, 1]$. In comparison, commitment in the sequential game increases A's utility by 21%.

The concept of commitment has been observed in the domain of auction design, even though sometimes implicitly. Note that early bidding and sniping in online auctions (e.g., eBay auctions) can be regarded as two forms of commitment (Roth and Ockenfels 2002; Gray and Reiley 2013). An advertiser that has a "passive" image (i.e., rarely changes the bid, or always submits

[2] We will use a first-price auction as a running example throughout this paper. We extend the results to all-pay auctions and general rank-and-bid based auctions in Section 7.

low bids) in sponsored search auctions can be seen as another form of commitment. Abraham et al. (2013) consider a super bidder who has access to more information than others and study how this will affect the auctioneer's revenue in a solution concept called tremble-robust equilibrium. Their setting is similar but different from this study. There is a leader in both studies, but in our setting, the leader does not have any informational advantage. Skreta (2006) considers another type of commitment where the auctioneer is lacking in credibility to reserve the item and studies how this lack of commitment affects revenue.

A closely-related parallel work is Xu and Ligett (2014). They characterize optimal commitment for first-price auctions with *complete* information. For the case with incomplete information, they assume that the bidders' types are drawn from discrete distributions and prove a partial property that the commitment function can be divided into pieces. In comparison, we consider general continuous distributions and obtain closed-form characterizations for a more general class of auctions. Our technical approach also offers new insights in solving such models.

In summary, we study, in this paper with a Bayesian Stackelberg game of incomplete information, the optimal Bayesian commitment in first-price auctions. There is one item and two bidders A and B. The two bidders' valuation are independent and differentiable distributions $F_1(x)$ and $F_2(y)$, with density functions $f_1(x)$ and $f_2(x)$. We later consider the commitment problem in more general *rank-and-bid based auctions* (Chawla and Hartline 2013).

1.2. Our contributions

Our main contribution is a general approach to solve and characterize optimal Bayesian commitment in this class of auctions, for any continuous-type distributions. In particular, applying our approach, we are able to examine optimal commitments for first-price and all-pay auctions in *closed-form* for fairly general distributions. Our modeling approach offers a way to solve several nontrivial technical challenges. The modeling innovations mitigate the difficulties of deriving a game-theoretical prediction in first-price and all-pay auctions with asymmetric-type distributions. We dedicate Section 3 to introduce the technical contribution. Next, we focus on the economic interpretations of our results.

1.2.1. Equal-bid function and characterization We define equal-bid function g as a mapping function from leader A's value to follower B's value. $g(x)$ is the minimum of the follower's value which has the best response equal to leader A's bid $s_A(x)$. That is, when A's value is x, we can find B's value $g(x)$, such that A and B's bids are both $s_A(x)$. When multiple B's values satisfy this mapping, $g(x)$ is the lowest. We can prove that g is well defined and is a left-continuous function. This equal-bid function g allows us to derive our main theorem in a tractable way.

Theorem 5.7 gives the characterization of the optimal commitment strategy in first-price auctions:

$$s_A^*(x) = \frac{1}{F_1[x]} \int_{a_1}^{x} f_1(t)g(t)dt.$$

With the equal-bid function, we can conveniently simplify the model and represent equations with function g. For example, the winning probability of leader A with value x can be expressed as $F_2[g(x)]$. We therefore can convert the problem to be finding the corresponding optimal equal-bid function.

1.2.2. Structure and closed-form Theorem 6.2 suggests when F_2 is uniformly distributed on $[b_1, b_2]$,

- if $\forall t, 2f_1^2(t) - F_1[t]f_1'(t) \geq 0$, then optimal $g(x)$ has at most 3 values.
- if $b_1 = 0$, then $g(x)$ has at most 2 values.

Theorem 6.3 gives the closed-form solution of the first-price auction when the two bidders' types are drawn uniformly from $[0, 1]$.

$$s_A^*(x) = \begin{cases} 0 & x \in [0, t_0] \\ 1 - \frac{t_0}{x} & x \in (t_0, 1] \end{cases} \quad , t_0 \approx 0.567$$

In this special case, as depicted in Figure 1, the optimal profit of the leader

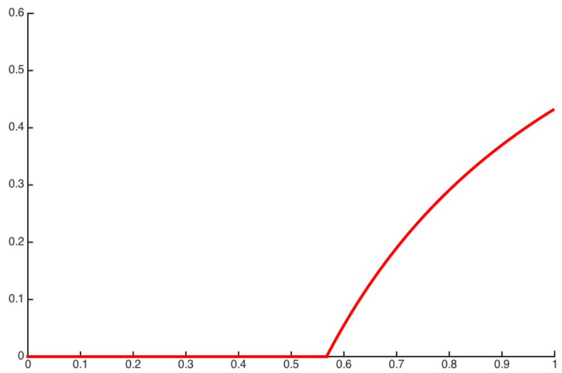

FIGURE 1.— The leader's optimal commitment strategy keeps zero when his type is low. It increases rapidly when the type is large enough.

is 0.22, an increase of 32% compared to the profit obtainable in a symmetric Bayesian Nash Equilibrium. Figure 1's horizontal axis is leader's type, and the vertical axis is his strategy. This special case reveals an interesting observation: The leader bids very passively when his type is low. Even worse, depending on type distributions, he bids 0 when his type is below a threshold. This result stands in stark contrast to the intuition that in first-price and all-pay auctions bidding 0 has no chance of winning at all. However, a closer scrutiny suggests otherwise: by committing to a passive image, the leader credibly ensures the follower that he has no intention to compete when he has a low type, thus effectively brings down the follower's bid. As a result, the leader earns extra surplus in the auction with less competition when he has a high type. We also note that such passive bidding behaviors had been observed in major search engines such as Yahoo and Baidu (before they switch to GSP).

Furthermore, the commitment solutions are largely consistent with the collusion or collusive-seeming behavior studied in first-price auctions (McAfee and McMillan 1992; Marshall and Marx 2007; Aryal and Gabrielli 2013; Lopomo et al. 2011; Pesendorfer 2000): Players coordinate to bring down the prices. Our results further suggest that such collusive behaviors are stable: the trust between the players is built on (1) the leader's credibility to commit, and (2) the follower's rationality.

1.2.3. All-pay auctions Theorem 7.8 gives the optimal strategy in all-pay auctions:

$$s_A^*(x) = \int_{a_1}^{x} f_1(t)g(t)dt$$

Theorem 7.9 suggests when f_2 is weakly increasing, then optimal $g(x)$ is a step function consisting of at most 2 values, 0 and b_2, and the cut point t_0 of g is the solution of $b_2 - t - b_2F_1[t] = 0$.

$$s_A^*(x) = \begin{cases} 0 & x \in [a_1, t_0] \\ b_2(F_1[x] - F_1[t_0]) & x \in (t_0, a_2] \end{cases}$$

2. The Setting

We consider a single item auction with two bidders, one called the *leader* A (male) and the other called the *follower* B (female). Bidder A has a private valuation x drawn from distribution F_1 with support $[a_1, a_2]$, while bidder B's private valuation y is drawn from distribution F_2 with support on $[b_1, b_2]$. We use f_1 and f_2 to denote the density functions of F_1 and F_2, respectively. We also write $A = x$ to denote the case where A's type is x. Similarly, we can write $B = y$.

Leader A commits to a Bayesian strategy $s_A : [a_1, a_2] \to \triangle R$, where $\triangle R$ denotes the set of bid distributions on R. He announces this strategy and the follower B best responds to the leader's committed strategy via a single bid.[3] We follow the standard definition by Conitzer and Sandholm (2006) of Bayesian commitment that the leader only announces his strategy, the function $s_A(\cdot)$, without revealing his actual type. Compared to the utility from BNE, being able to commit increases the leader's utility.

The timing of the game is summarized as follows:

1. Leader A announces her Bayesian strategy $s_A(\cdot)$ to follower B;
2. Leader A then draws a random type x from a type distribution F_1, which is commonly known;
3. Leader A and B participate in a simultaneously auction, where A commits to bid $s_A(x)$, while B knows $s_A(\cdot)$ but not x and bids optimally to her available information.[4]
4. The payoffs of both players are then determined according to the auction rule.

There are two tie-breaking rules.

Assumption 2.1 *When B has multiple best responses, she will choose the one that maximizes A's winning probability.*

Assumption 2.2 *When there is a tie, the good will be assigned to B.*

We make these assumptions for expositional simplicity. Our main results do not depend on these assumptions. We will show formally in Appendix C that neither of the assumptions is necessary.

Given s_A, B's best response is fixed by assumption, and both players' winning probabilities are determined.

Our goal is to solve for the optimal s_A in a general class of auctions called the *rank-and-bid based auctions* (Chawla and Hartline 2013), in which bidders' payments are decided by their own bids and ranks. First price

[3]It is easy to see that it is never profitable for B to use a mixed strategy.

[4]We use first-price auctions as a running example, these results remain to be valid for all-pay auctions.

auctions and all-pay auctions belong to this class of auctions.

Finding the optimal strategy s_A is known to be difficult. Conitzer and Sandholm (2006) show that computing optimal commitments in general Bayesian games is NP-hard.

Definition 2.3 *$P_A[x, s_A]$ denotes A's winning probability when he has type x and adopts strategy s_A, while $P_B[t, s_A]$ is B's winning probability when bidding t against A's strategy s_A. In circumstances where there is no ambiguity, we use $P_A[t], P_B[t]$ instead.*

Definition 2.4 *We use u_A and u_B to denote A and B's expected utility, respectively. Given s_A, we can derive P_B. Since B best responses to A, we have*

$$u_B(y) = \sup_{t \geq 0}(y \cdot P_B[t] - t \cdot P_B[t])$$

where t represents B's bid. After figuring out B's response, we can compute A's expected utility,

$$u_A(s_A) = E_{x \sim F_1}(x \cdot P_A[x] - s_A(x) \cdot P_A[x])$$

Although B's valuation is on $[b_1, b_2]$, we extend its definition domain to $[0, b_2]$, and let $F_2[x] = 0$, for $x < b_1$.

Note that the follower cares about the local maximal utility on each type, and the input for u_B is a type. The leader cares about the maximal expected utility on the whole range and the input for u_A is a strategy function.

We use sup rather than max here because we need to show that max is attainable, which is done in Lemma 4.1. We also note that bidding zero yields a nonnegative utility for B, so $u_B \geq 0$ and $u_B(0) = 0$.

Definition 2.5 *Let $S_B(y, s_A)$ denote $B = y$'s best responses against A's strategy s_A. In circumstances where there is no ambiguity, we use $S_B(y)$ instead.*

We will prove this definition is well-defined in Lemma 4.1, i.e. $S_B(y)$ is nonempty.

3. An overview of our approach

The main contribution of this paper is to put forward a general approach for solving optimal Bayesian commitments in rank-and-bid based auctions. The approach can be sketched as follows. It is useful to understand the intuition behind this approach before we proceed to the details.

3.1. *Difficulties of the problem*

If the optimal commitment was monotone and differentiable, we can use the first-order condition to get the result: follower best responses to the leader. That is, (and if $g(x)$ is well-defined)

$$(g(x) - s_A(t))F_1[t] \text{ is maximized at } t = x$$
$$(g(x) - s_A(t))f_1(t) - s'_A(t)F_1[t] = 0$$
$$g(x)f_1(t) = [s_A(t)F_1[t]]'$$
$$s_A(x) = \tfrac{1}{F_1(x)} \int_{a_1}^{x} g(t)f_1(t)dt$$

However, Maskin and Riley (2003)'s approach to proving that the players' strategies are differentiable in a Nash equilibrium does not apply here. Their result is based on the fact that everyone best responds to others. In our setting, the leader's optimal strategy is not necessarily continuous and differentiable. The main difficulty of the problem is that the follower's best response cannot be represented as a closed-form function of the leader's Bayesian strategy, which is also a function. This difficulty further prevents one from obtaining a closed-form representation of the leader's winning probability and utility. As a result, standard functional optimization techniques cannot be applied.

To appreciate the difficulty, it is helpful to look at the problem of finding a BNE with two asymmetric bidders in a first-price auction — one of the most elusive open problems in the analysis of auctions (Kaplan and Zamir

2012; Hartline et al. 2014). The main barrier in this literature is exactly the difficulty to represent one's best response as a concise function of the other's strategy.

Previous work on this problem focuses on different cases of finding optimal commitments in first price auctions with complete information (e.g., Xu and Ligett 2014). This stream of work also examines the Bayesian case when the follower's type is drawn from a discrete distribution. So far, this literature establishes some partial properties of the optimal commitment problem in the discrete case. The approach and results cannot be extended to our continuous case.

3.2. *Step one: sorting the leader's strategy and making it monotone*

As discussed above, one of the obstacles is that the optimal leader strategy may not be monotone or differentiable. Our first effort is to sort the leader's strategy. We prove that, for any leader's strategy (optimal or not), one can sort it into a monotone function that preserves the follower's best response without hurting the leader's utility. In other words, in this step we can prove that the leader always has an equivalent monotone strategy.

3.3. *Step two: smoothing the leader's strategy and making it continuous*

A more difficult task is to smooth the leader's strategy, that is, to turn it into a continuous and differentiable function. To achieve this goal, we introduce a methodological innovation called *equal-utility curve*. Roughly, given the follower's type, the equal-utility curve represents a leader's strategy such that the follower gets the same utility no matter what she bids. We can show that such a curve always exists. Furthermore, the supremum (over all follower's types) of all such curves defines a new leader's strategy that enjoys the following important properties: it is continuous, left and right differentiable, preserves the follower's best response, and weakly improves the leader's utility. To this point, one can truly focus on monotone, continuously differentiable leader strategies. Actually after this step, the modified strategy has a nice structure.

3.4. Step three: representing the strategy by a differentiable equal-bid function

A key insight in this step is to represent everything (the strategies, utilities and winning probabilities of both players) as a function of some g, coined the *equal-bid function*, that maps a leader's type to a follower's type. Intuitively, $g(t)$ is the follower's type at which she submits *the same* bid as the leader does when the leader has type t. In other words, $g(t)$, later to be proved as monotone, can be seen as a bifurcation type between winning and losing for the follower. As one can imagine, together with the cumulative distribution function of the follower's type, we can represent the leader's winning probability (hence utility) as a function of g. A similar but different idea has appeared in Hafalir and Krishna (2008) and Lebrun (1999) in which *inverse bid functions* are used to represent the best responses of players. However, *inverse bid functions* are in pairs and the two functions make the final optimization problem complicated. In contrast, our proposed equal-bid function represents everything with a single function.

3.5. Step four: optimizing A's expected utility in terms of equal-bid function

With the above transformations, it turns out that we can find a bijection between the set of monotone, continuously differentiable leader's strategies and the set of continuous and monotone equal-bid functions. As a result, we can focus on optimizing over equal-bid functions and the Lagrangian method applies.

We conclude with a characterization of the optimal Bayesian commitment for general follower type distributions and compute the closed-form optimal commitments for both first-price and all-pay auctions when the follower has uniform type distributions.

4. Sorting and smoothing the leader's strategy

We first transform an arbitrary leader strategy into a continuous weakly increasing strategy and show that this transformation does not hurt the leader's expected utility.

In the next section, we prove some additional properties, such as differentiability, for the transformed strategy.

We first show that the notion of "best response" is well defined for the follower B.

Lemma 4.1 *For any B's valuation y, B has a best response, i.e. $u_B(y)$ can be attained by some bid and the lowest bid exists among the best responses.*

Proof. All proofs are in the appendix. □

By Assumption 2.2, B always chooses the lowest bid among all best responses.

4.1. Sorting s_A

For an arbitrary strategy s_A, the support of $s_A(v)$ on value v may not be a single bid. Function s_A could also be non-monotone. The following lemma suggests, to find the optimal Bayesian commitment, it suffices to consider the strategies with the desirable properties below.

Lemma 4.2 *We can sort any strategy s_A for A into a new strategy function $\breve{s}_A$ such that (1) $\breve{s}_A(v)$ is a deterministic bid for any v and is weakly* *increasing in* v, *(2) the best response of* $\breve{s}_A$ *remains the same as* s_A, *and (3)* $\breve{s}_A$ *yields at least the same utility for the leader as* s_A.

For example, suppose leader A has equal probability on three types 1, 2 and 3. Then $\breve{s}_A$ brings higher expected utility than s_A.

$$s_A(x) = \begin{cases} 0.8 & x = 1 \\ 0.6 & x = 2 \\ 1 & x = 3 \end{cases} \qquad \breve{s}_A(x) = \begin{cases} 0.6 & x = 1 \\ 0.8 & x = 2 \\ 1 & x = 3 \end{cases}$$

With this result, for ease of presentation, we can use s_A to denote $\breve{s}_A$ henceforth. So s_A is a weakly increasing, nonnegative strategy function.

The following example shows how to calculate u_B in a first-price auction.

Example 4.3 *Both bidders' value distribution are uniform on [0,1].*

$$s_A(x) = \begin{cases} x/4 & x \leq 0.4 \\ x - 0.3 & x > 0.4 \end{cases}$$

By definition $u_B(y) = \max\{\max_{t\leq 0.1}(y-t)\cdot 4t, \max_{t>0.1}(y-t)(t+0.3)\}$, *we have*

$$u_B(y) = \begin{cases} y^2 & y \in [0, 0.2) \\ 0.4y - 0.04 & y \in [0.2, 0.5] \\ (y+0.3)^2/4 & y \in (0.5, 1] \end{cases}$$

The following Lemma suggests the set of best responses of B at type y_1 generally does not intersect with the set of best responses at type y_2 and these sets are well sorted by the value of y.

Lemma 4.4 *For B's valuations* $y_1 < y_2$, *if* $\exists a \in S_B(y_1), b \in S_B(y_2)$ *and* $a > b$, *then we have* $S_B(y_1) \subseteq S_B(y_2)$, $u_B(y_1) = u_B(y_2) = 0$ *and* $P_B[b] = P_B[a] = 0$.

We can now prove that the follower's utility is continuous and monotone.

Lemma 4.5 $u_B(y)$ *is continuous and weakly increasing.*

4.2. *Smoothing s_A*

To smooth s_A into a continuous and differentiable function, we now introduce an important innovation of our approach: the equal-utility curve.

Definition 4.6 *Define equal-utility curve $eu_B(\cdot,\cdot) : [0, b_2] \times (a_1, a_2] \to \mathbb{R}$,*

$$eu_B(y,x) = y - \frac{u_B(y)}{F_1[x]} \tag{1}$$

The interpretation of $eu_B(y,\cdot)$ is that, any value of $eu_B(y,\cdot)$ (as a function of x) is a best response of the follower at type y.

Consider Example 4.3, in a first-price auction, $F_1[x] = F_2[x] = x$, $\forall x \in [0,1]$, the definition of eu_B above is simplified as

$$eu_B(y,x) = y - \frac{u_B(y)}{x}.$$

When B's value $y = 0.5$, her best response is to bid 0.1 with utility 0.16. So $u_B(0.5) = 0.16$, and the equal-utility curve is $eu_B(0.5, x) = 0.5 - \frac{0.16}{x}$, shown in Figure 2. If A uses strategy

$$\max\{eu_B(0.5,x), 0\} = \begin{cases} 0 & x \in [0, 0.32) \\ 0.5 - \frac{0.16}{x} & x \in [0.32, 1] \end{cases}$$

then the utility of B when $y = 0.5$ is the same for any bid in $[0, 0.34]$.

Function eu_B represents the leader's strategy against which the follower will achieve the same largest utility no matter what the follower bids. It's easy to check that $eu_B(0,\cdot) = 0$.

Lemma 4.7 *(1) $eu_B(y,x)$ is weakly increasing and differentiable in x. $eu_B(y,x)$ is continuous in y.*
(2) $eu_B(y,x) \le s_A(x)$

We are now ready to introduce the smoothing method, by constructing an envelope s^*_A using eu_B.

Definition 4.8 $s^*_A(x) = \sup_{y\in[0,b_2]} eu_B(y,x), \forall x \in (a_1, a_2]$

We will prove that strategy s^*_A yields at least the same revenue for A as strategy s_A. The outline is as follows. First, we prove that, although the

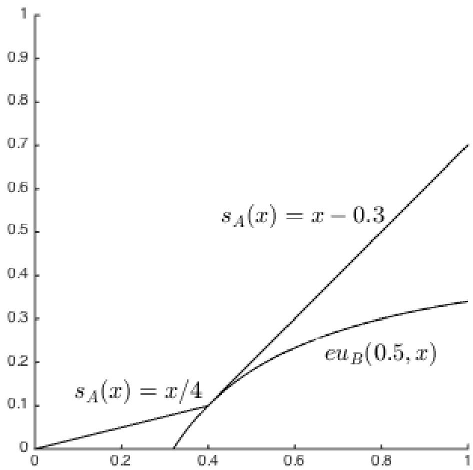

FIGURE 2.— Equal-utility Curve. When $y = 0.5$, the utility of bidding 0 or 0.3 are always 0.16.

leader's bid distribution changes, we keep the utility of the follower to be the same. Second, we prove the best response of B is still the best response after smoothing, for any follower's value (Lemma 4.10). Third, we prove the leader's winning probability does not change. Finally, since the leader's bid is weakly decreasing, we can prove that the leader's utility weakly increases after smoothing (Theorem 4.14).

A mathematical view of the motivation of $s_A^*(x)$ can be found in Remark 2. The idea is to suppress the bids of the leader while maintain his winning probability.

Consider Example 4.3, we can find $s_A^*(x)$ easily:

$$s_A^*(x) = \begin{cases} x/4 & x \in [0, 0.4) \\ x - 0.3 & x \in [0.4, 0.65] \\ 1 - \frac{0.65^2}{x} & x \in (0.65, 1] \end{cases}.$$

We now prove some basic properties of s_A^* that will be used later.

Lemma 4.9 *(1) For any* $x \in (a_1, a_2]$, $(x, s_A^*(x))$ *must lie on some Equal-Utility Curve.*

(2) When $s_A^*(x) > \lim_{t\to a_1} s_A^*(t)$, $s_A^*(x)$ *strictly increases.*

(3) When $s_A^*(x) = \lim_{t\to a_1} s_A^*(t)$, $(x, s_A^*(x))$ *lies on* $eu_B(s_A^*(x), \cdot)$, *and* $u_B(s_A^*(x)) = 0$.

(4) For any x, *we have* $s_A^*(x) \le s_A(x)$.

(5) $s_A^*(x)$ *is continuous.*[5]

Up to now, the domain of s_A^* is defined as $(a_1, a_2]$. Since s_A^* is continuous, we can define $s_A^*(a_1) = \lim_{x\to a_1} s_A^*(x)$. For simplicity, we use $P_B^*[t]$ and $S_B^*(y)$ instead of $P_B[t, s_A^*]$ and $S_B(y, s_A^*)$. Next, we study how the follower's utility and best response would change in s_A and s_A^*.

Lemma 4.10 *(1) When A's strategy* s_A *is changed to* s_A^*, $u_B(y)$ *remains the same.*

(2) If $t \in S_B(y)$ *then* $t \in S_B^*(y)$, $S_B(y) \subseteq S_B^*(y)$. *If* $P_B^*[t] \neq P_B[t]$ *then* $u_B(y) = 0$ *and* $t = \lim_{x\to a_1} s_A^*(x)$

The lemma below draws connections between equal-utility curve and s_A^*.

Lemma 4.11 *(1) If* $(x_0, s_A^*(x_0))$ *lies on equal-utility line* $eu_B(y_0, \cdot)$, *then* $s_A^*(x_0) \in S_B^*(y_0)$. *(2) If* $s_A^*(x_0) \in S_B^*(y_0)$ *and* $s_A^*(x_0) \neq \lim_{x\to a_1} s_A^*(x)$ *then* $(x_0, s_A^*(x_0))$ *lies on equal-utility line* $eu_B(y_0, \cdot)$.

The best response set S_B^* of the follower in s_A^* is a superset of S_B. Since the best response of the follower is sorted, $S_B^*(y)$ is bounded by any element in $S_B^*(y-\epsilon)$ and $S_B^*(y+\epsilon)$. Therefore, it is bounded by $S_B(y-\epsilon)$ and $S_B(y+\epsilon)$. We then prove that for most of the types, the winning probability of the leader will not change.

Definition 4.12 *If* $\exists x$ *such that* $s_A^*(x) = \lim_{t\to a_1} s_A^*(t)$, *let* $\hat{x} = \sup\{x | \lim_{t\to a_1} s_A^*(t) = s_A^*(x)\}$.

[5]The limit of a continuous function may not be continuous, so this argument is not trivial. Consider $y_k(x) = kx, x \in [0, 1/k]$, constantly zero for $x \le 0$ and constant one for $x \ge 1/k$. Clearly, y_k is continuous but $sup_k y_k$ is not.

Since s_A^* is continuous, $s_A^*(\hat{x}) = \lim_{t \to a_1} s_A^*(t)$. The following Lemma tells what s_A is if s_A^* has a constant interval in the beginning. The proof is similar to that of Lemma 4.9.

Lemma 4.13 *If* $\exists x$ *such that* $s_A^*(x) = \lim_{t \to a_1} s_A^*(t)$*, we have* $s_A^*(x) = s_A^*(\hat{x})$ $\forall x < \hat{x}$*.*

Combined with the fact that the bids decrease in s_A^* (thus lower payment) and the winning probability remains the same, we prove that the expected utility does not decrease.

Theorem 4.14 *By using* s_A^* *instead of* s_A*, the expected utility of* A *does not decrease.*

The intuition is that both strategies lead to the same follower's best response, while s_A^* is lower and the leader has less payment. It may be possible that s_A^* is worse at countable number of breaking points, however, the effect of these points to the expected utility is 0.

5. Bijective mapping between s_A^* and g

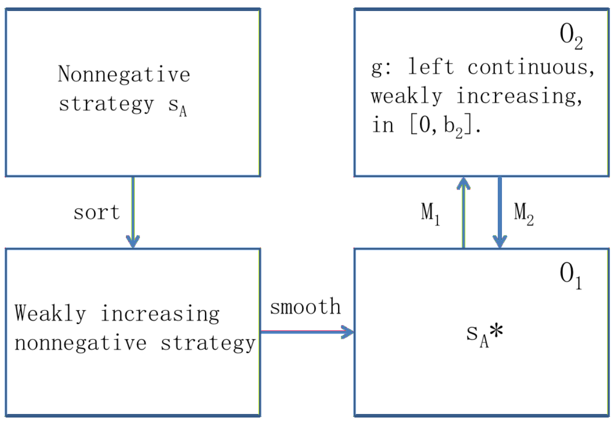

FIGURE 3.— M_1 is bijective between O_1 and O_2

The final step is to show that every s_A^* can be represented by a function g. Once we obtain such a function, we will be able to focus on optimizing such a g instead. Figure 3 shows the bijective mapping in this section.

Definition 5.1 $\hat{y} = \sup\{y | u_B(y) = 0\}$

Definition 5.2 $\forall x > a_1$, $Y(x) = \{y | eu_B(y, \cdot)$ *passes through point* $(x, s_A^*(x))\}$.

Lemma 5.3 *(1)* $Y(x)$ *is closed. (2)* $Y(x) \geq \hat{y}$. *(3) For all* $x_1 < x_2$, $Y(x_1) \leq Y(x_2)$. *(4)* $(\hat{y}, b_2] \subseteq \cup_x Y(x)$ *(5)* $Y(x)$ *is an interval or is a unique number.* $Y(x)$ *contains only one element for almost all* x.

Definition 5.4 *Equal-bid function* $g(x) = \min Y(x), \forall x \in (a_1, a_2]$

The intuitive explanation of g is: when the follower's type $y = g(x)$, one of her best response is equal to the bid $s_A^*(x)$ by the leader. In most of the time except for countable types, if the follower gives the same bid as the leader, then the lowest type of the follower must be $g(x)$. In other words, the follower with value $g(x)$ submits the same bid as the leader who has value x.

With equal-bid function, the winning probability of the leader has a surprisingly concise form, as shown in Lemma 5.5.

Consider Example 4.3, we have

$$g(x) = \begin{cases} x/2 & x \in [0, 0.4) \\ 2x - 0.3 & x \in [0.4, 0.65] \\ 1 & x \in (0.65, 1] \end{cases}$$

It's easy to check that $s_A(0.45) = 0.15$, $g(0.45) = 0.6$, $S_B(0.6) = \{0.15\}$. The leader with value 0.45 bids 0.15, same as the follower with value $g(0.45) = 0.6$.

By lemma 5.3, $\{y | eu_B(y, x) = s_A^*(x)\}$ is closed, so $g(x)$ is well defined. When $y < g(x)$, the leader $A = x$ beats the follower y by Lemma 4.4. When $y \geq g(x)$, the follower y beats the leader $A = x$ by the tie-breaking rule. So we can calculate the winning probability of the leader $A = x$ using $g(x)$.

Lemma 5.5 *Using strategy* s_A^**, the winning probability of* A *with type* x *is* $F_2[g(x)]$.

Lemma 5.6 *(1) $g(x)$ weakly increases. (2) $g(x)$ is left continuous.*

Note $s_A^*(x)$ is not yet defined on a_1. In fact, bidding with zero probability does not affect overall utility. We can define $s_A^*(a_1) = \lim_{t\to a_1} s_A^*(t)$ for convenience.

Now we prove the last a few desirable properties of s_A^*: s_A^* is differentiable on both sides. Based on the derivatives, we find the relationship between g and s_A^*.

Theorem 5.7

1. *$s_A^*(x)$ is left-hand differentiable and right-hand differentiable.*
2. *$s_A^*(x) = \frac{1}{F_1[x]}\int_{a_1}^x f_1(t)g(t)dt$.*

Remark 1. *Up to now, we have developed a new strategy s_A^* for A based on s_A, with at least 2 desirable properties: it yields at least as much utility as s_A and is left-hand differentiable and right-hand differentiable. In the following, we will calculate the winning probability and find out the s_A^* with the optimal utility.*

Remark 2. *From s_A (we only need the weakly increasing condition), we can define g directly, but s_A cannot be calculated by g. To see this, the follower bids $s_A(t)$ and achieves the highest utility when $t = x$. Taking first-price auctions for example, we have $(g(x) - s_A(x))F_1[x] \geq (g(x) - s_A(t))F_1[t]\forall t$, then $s_A(t) \geq g(x) - \frac{(g(x)-s_A(x))F_1[x]}{F_1[t]}$, equality can be achieved by setting t to be x. Moreover $s_A(t) \geq \sup_x(g(x) - \frac{u_B(x)}{F_1[t]}) = \sup_x eu_B(g(x), t)$, equality may not be achieved, because if we fix t first, there might be no corresponding x. Thus we do not have the exact formula of $s_A(t)$. If s_A is optimal, for any leader's type, his bid should be as small as possible without changing the follower's behavior. To do this, when $s_A(t) > \sup_x eu_B(g(x), t)$, we can decrease his bid to $\epsilon + \sup_x eu_B(g(x), t)$, without letting the follower match his new bid. This is the nature of the smoothing method. So in the optimal strategy, we should have $s_A(t) = \sup_x eu_B(g(x), t)$, which is exactly the new strategy out of the smoothing method, s_A^*.*

We can check the correctness of relationship between g and s_A^* in Example 4.2.

When $x_0 \in [0, 0.4]$,

$$\frac{1}{F_1[x_0]}\int_{a_1}^{x_0} g(x)f_1(x)dx = \frac{1}{x_0}[\int_0^{x_0} x/2dx] = x_0/4 = s_A^*(x_0)$$

When $x_0 \in [0.4, 0.65]$,

$$\frac{1}{F_1[x_0]}\int_{a_1}^{x_0} g(x)f_1(x)dx = \frac{1}{x_0}[\int_0^{0.4} x/2dx+\int_{0.4}^{x_0}(2x-0.3)dx] = x_0-0.3 = s_A^*(x_0)$$

When $x_0 \in [0.65, 1]$,

$$\frac{1}{F_1[x_0]}\int_{a_1}^{x_0} g(x)f_1(x)dx = \frac{1}{x_0}[\int_0^{0.4} x/2dx + \int_{0.4}^{0.65}(2x-0.3)dx + \int_{0.65}^{x_0} 1dx]$$
$$= \frac{1}{x_0}[0.04 + 0.65 * 0.35 - 0.04 + x_0 - 0.65] = 1 - \frac{0.65^2}{x} = s_A^*(x_0)$$

Definition 5.8

$$O_1 = \{s_A| \textit{ strategies resulted from any nonnegative strategy after smoothing}\}$$
$$O_2 = \{(g, s_A(a_1))|g \textit{ is weakly increasing and left continuous and in } [0, b_2]\}$$

Since the domain of y in $eu_B(y, x)$ is $[0, b_2]$, we have $Y(x) \subset [0, b_2]$ and $g(x) \in [0, b_2]$. Thus Definition 5.4 gives a mapping $M_1 : O_1 \to O_2$. In fact, we will prove that there is a bijective mapping between the two sets. The idea is that we construct a mapping $M_2 : O_2 \to O_1$ and prove $M_1 \circ M_2 = I$.

Theorem 5.9 *There is a bijective mapping between O_1 and O_2.*

Thus finding the optimal strategy is equivalent to finding the optimal function g such that $(g, 0) \in O_2$.

6. Optimizing equal-bid function g

In this section, we solve for the optimal g in order to derive the final form of s_A^*.

6.1. General optimization

Since $s_A^*(x) = \frac{1}{F_1[x]}\int_{a_1}^x f_1(t)g(t)dt$, the expected utility becomes

$$u_A(s_A^*) = \int_{a_1}^{a_2}[x - \frac{1}{F_1[x]}\int_{a_1}^x f_1(t)g(t)dt]F_2[g(x)]f_1(x)dx$$

This is a function of g, denoted by $M(g)$. For any admissible function $g+\epsilon j$, i.e. $(g+\epsilon j, 0) \in O_2$, we have $M(g) \geq M(g+\epsilon j)$. Consider the marginal loss in direction j, we have

$$\begin{aligned}
0 &\geq \lim_{\epsilon \to 0}\frac{M(g+\epsilon j) - M(g)}{\epsilon} \\
&= \int_{a_1}^{a_2}[x - \frac{1}{F_1[x]}\int_{a_1}^x f_1(t)g(t)dt]f_2(g(x))j(x)f_1(x)dx \\
&\quad + \int_{a_1}^{a_2} -\frac{1}{F_1[x]}\int_{a_1}^x f_1(t)g(t)dtF_2[g(x)]f_1(x)dx \\
&\overset{(2)}{=} \int_{a_1}^{a_2} j(x)f_1(x)[f_2(g(x))(x - \frac{1}{F_1[x]}\int_{a_1}^x f_1(t)g(t)dt) - \int_x^{a_2}\frac{F_2[g(t)]f_1(t)}{F_1[t]}dt]
\end{aligned}$$

Let $h(x)$ denote the coefficient of $j(x)$:

$$h(x) = f_1(x)(xf_2(g(x)) - \frac{f_2(g(x))}{F_1[x]}\int_{a_1}^x f_1(t)g(t)dt - \int_x^{a_2}\frac{F_2[g(t)]f_1(t)}{F_1[t]}dt)$$

We can deduce the optimal g when $h(x)$ has some good property.

Theorem 6.1 *(1) For an interval L, if $h(x) > 0,\ x \in L$, we have*

$$g(x) = \lim_{t \to (\sup L)^+} g(t)\ x \in L$$

Moreover, if $\sup L = a_2$, then $g(x) = b_2, x \in L$.

Similarly, if $h(x) < 0, x \in L$, we have

$$g(x) = \lim_{t \to \inf L} g(t), x \in L$$

If $\inf L = a_1$, then $g(x) = 0, x \in L$.

(2)There is an optimal g such that $g(x) \in 0 \cup [b_1, b_2]$.

In fact, g can be derived explicitly in fairly general settings, as we show below.

6.2. Some specific optimization results

Theorem 6.2 *When F_2 is uniformly distributed on $[b_1, b_2]$,*

(1) if $\forall t, 2f_1^2(t) - F_1[t]f_1'(t) \geq 0$, then optimal $g(x)$ consists of at most 3 values.

(2) if $b_1 = 0$, then optimal $g(x)$ consists of 2 values. When $t_0 = a_2 \int_{t_0}^{a_2} \frac{f_1(t)}{F_1[t]} dt$ has a solution,

$$g(x) = \begin{cases} 0 & x \in (a_1, t_0] \\ b_2 & x \in (t_0, a_2) \end{cases} \quad, \text{where } t_0 = a_2 \int_{t_0}^{a_2} \frac{f_1(t)}{F_1[t]} dt$$

Otherwise, $g(x) = b_2, \forall x \in [a_1, a_2]$.

Theorem 6.3 *In Example 4.3, the optimal utility of the leader is 0.22. The closed-form representation of the optimal $g(x)$ and $s_A^*(x)$ are:*

$$g(x) = \begin{cases} 0 & x \in [0, t_0] \\ 1 & x \in (t_0, 1] \end{cases} \qquad s_A^*(x) = \begin{cases} 0 & x \in [0, t_0] \\ 1 - \frac{t_0}{x} & x \in (t_0, 1] \end{cases} \quad, t_0 \approx 0.567$$

Here t_0 is the solution of $t_0 = b_2 \int_{t_0}^{a_2} \frac{f_1(x)}{F_1[x]} dx$.

We should note that the leader bids zero 56.7% of the time. The leader gives away positive utility when his value is low, i.e., when $x \in [0, 0.567]$. Although the probability of the leader's value lying in $[0, 0.567]$ is more than a half, he only gains a small amount of utility if he is in this case, since his valuation is low. When the leader has relatively high value, i.e. $x \in (0.567, 1]$, he wins with relatively lower bids. In equilibrium, the utility gains on this interval is high enough to compensate for the losses on the lower interval.

7. General rank-and-bid based auctions

Let us formally define *rank-and-bid based auctions.*

Definition 7.1 Rank-and-bid based auctions follow the following rules:

- *Allocation rule: The item is always allocated to the highest bidder.*
- *Payment rule: A bidder's payment depends only on his/her own bid and whether it wins or not. The payment is $p(t) = p^p(t)$ if the bidder loses with bid t. The payment is $p(t) = p^p(t) + p^w(t)$ if bidder wins with bid t. Here p^p and p^w are differentiable functions representing the agent's payments for participation and winning respectively.*

Clearly, first-price auctions and all-pay auctions belong to this class. For first-price auctions, $p^p(t) = 0$ and $p^w(t) = t$; for all-pay auctions, $p^p(t) = t$ and $p^w(t) = 0$.

Let $p^p(t) + p^w(t)$ be strictly increasing, $p^p(0) = p^w(0) = 0$, and assume p^p, p^w are differentiable functions. Again, the two auction formats above satisfy these assumptions. Note that A never submits any negative bid, we can assume $p^w, p^p : R \to R$ for expositional reasons.

When we consider the general case of rank-and-bid based auctions, some definitions and theorems need modification. Definition 2.4 is revised as follows

Definition 7.2 *We use u_A and u_B to denote A's and B's expected utility respectively.*

$$u_A(s_A) = E_{x\sim F_1}(x \cdot P_A[x, s_A] - p^p(s_A(t)) - p^w(s_A(t)) \cdot P_A[x, s_A])$$

while u_B can be conveniently expressed as,

$$u_B(y) = \sup_{t\geq 0}(y \cdot P_B[t] - p^p(t) - p^w(t) \cdot P_B[t])$$

i.e. $u_B(y)$ is the highest utility B can achieve with value y. Although B's valuation is on $[b_1, b_2]$, we extend its definition to $[0, b_2]$, and let $F_2[x] = 0$, for $x < b_1$.

Definition 4.6 can also be revised accordingly:

Definition 7.3 *Define equal-utility curve $eu_B(\cdot,\cdot)$: $[0, b_2] \times (a_1, a_2] \to \mathbb{R}$, such that $eu_B(y, x)$ is the only solution (solve for t) of*

$$u_B(y) = F_1[x]y - p^w(t)F_1[x] - p^p(t) \;\; x \in (a_1, a_2] \tag{3}$$

The following lemma proves that the equal-utility curve is weakly increasing and differentiable. It is also used in the proof of Theorem 5.9.

Lemma 7.4 *Given $a > 0$, $(p^w)'$ and $(p^p)'$ are nonnegative, $(p^w)' + (p^p)' > 0$. If $t(a,b)$ satisfies*

$$b + ap^w(t) + p^p(t) = 0$$

we can prove $t(a,b)$ exists, $\frac{\partial t}{\partial a}(a,b)$ and $\frac{\partial t}{\partial b}(a,b)$ are continuous. In particular

$$\frac{\partial t}{\partial a}(a,b) = \frac{-p^w(t)}{a(p^w)'(t) + (p^p)'(t)} \qquad \frac{\partial t}{\partial b}(a,b) = \frac{-1}{a(p^w)'(t) + (p^p)'(t)}$$

Lemma 4.9 is now revised as follows,

Lemma 7.5 *(1) For any $x \in (a_1, a_2]$, $(x, s_A^*(x))$ must lie on some equal-utility curve.*

(2) When $s_A^*(x) > \lim_{t \to a_1} s_A^*(t)$, $s_A^*(x)$ strictly increases.

(3) When $s_A^*(x) = \lim_{t \to a_1} s_A^*(t)$, $(x, s_A^*(x))$ lies on $eu_B(p^w(s_A^*(x)), \cdot)$, and $u_B(p^w(s_A^*(x))) = 0$, $p^p(p^w(s_A^*(x))) = 0$.

(4) For any x, we have $s_A^*(x) \le s_A(x)$.

(5) $s_A^*(x)$ is continuous.

Theorem 5.7 is changed as follows,

Theorem 7.6 *Assume both $(p^w)'$ and $(p^p)'$ are continuous, then*

1. *$s_A^*(x)$ is left-hand differentiable and right-hand differentiable.*
2. *s_A^* can be solved from $\int_{a_1}^{x} f_1(t)g(t)dt = p^w(s_A^*(x))F_1[x] + p^p(s_A^*(x)) - p^p(s_A^*(a_1))$.*

When p^p is not constantly zero, Definition 5.8 is changed to

Definition 7.7

$$O_1 = \{s_A |\ \text{strategies resulted from any nonnegative strategy after smoothing, } s_A(a_1) = 0\}$$
$$O_2 = \{(g, s_A(a_1)) | g \text{ is weakly increasing, left continuous and defined on the interval } [0, b_2]\}$$

7.1. *All-pay auction*

In all-pay auctions, $p^w(t) = 0$ and $p^p(t) = t$. The optimal commitment strategy is as follows,

Theorem 7.8 *In the optimal strategy,* $s_A^*(a_1) = 0$,

$$s_A^*(x) = \int_{a_1}^{x} f_1(t)g(t)dt.$$

Correspondingly, the coefficient of j becomes $h(x) = f_1(x)[xf_2(g(x)) - 1 + F_1[x]]$. We next derive closed-form solutions for some specific cases.

Theorem 7.9 *When f_2 is weakly increasing, then the optimal $g(x)$ is a step function consisting of at most 2 values, 0 and b_2, and the cut point t_0 of g is the solution of $b_2 - t - b_2F_1[t] = 0$. In particular, when F_1 is uniformly distributed, $t_0 = \frac{b_2a_2}{b_2+a_2-a_1}$.*

$$g(x) = \begin{cases} 0 & x \in [a_1, t_0] \\ b_2 & x \in (t_0, a_2] \end{cases} \qquad s_A^*(x) = \begin{cases} 0 & x \in [a_1, t_0] \\ b_2(F_1[x] - F_1[t_0]) & x \in (t_0, a_2] \end{cases}$$

8. Conclusion

In many institutional settings, out of many participants, one dominant player may exert a significant influence on the potential equilibrium that will be played by all. Commitment is one such way. When a dominant player can establish a reputation, commitment to a publicly known strategy can result in higher profits for the player.

This paper offers a framework to derive the closed-form solution to a class of games including first-price and all-pay auctions when one player can use Bayesian commitment as a viable strategy. Different from works in the literature, our model allows for incomplete information with asymmetric value distributions.

To solve this problem, we propose some modeling innovations. Specifically, we propose a modeling concept called equal-bid function to build a bridge between two players' strategies. Another concept called equal-utility curve transforms any commitment strategy into a weakly better continuous and differentiable strategy.

Our findings offer some interesting insights. With Bayesian commitment, a relatively low-valued leader will be less aggressive so that the two sides can collude and reduce competition. The leader can enjoy high payoffs when he has a sufficiently high value. Overall, we show that Bayesian commitment allows the leader to obtain higher payoffs compared to the case when commitment is not allowed. Our methodological approach can be adopted in similar settings to find optimal commitment strategies.

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A. Appendix: Proofs

Proof of **Lemma 4.1**.

Suppose otherwise $u_B(y)$ cannot be attained. Since the domain is bounded, by the definition of $u_B(y)$, there exists t_0, such that:

$$\exists\{t_n\} \to t_0,\ s.t.\ \lim_{n\to\infty}[y \cdot P_B[t_n] - p^p(t_n) - p^w(t_n) \cdot P_B[t_n]] = u_B(y)$$

By tie-breaking rule, we have $\lim_{n\to\infty} P_B[t_n] \le P_B[t_0]$. Since $u_B(y) \ge 0$, we have $\lim_{n\to\infty}(y - t_n) \ge 0$.

$$u_B(y) = \lim[(y - t_n)P_B[t_n]] \le (y - t_0) \cdot P_B[t_0] \le u_B(y)$$

Hence, $u_B(y) = (y - t_0) \cdot P_B[t_0]$, that is $u_B(y)$ can be attained when bidding t_0.

Next we prove the lowest best response exists. Suppose otherwise there is no lowest best response, then among all the best responses, there exists $\underline{t}$, such that

$$\exists\{t_n\} \to \underline{t}\ s.t.\ u_B(y) = \lim y \cdot P_B[t_n] - t_n \cdot P_B[t_n] \quad \forall n$$

By same argument as above, we know

$$u_B(y) = y \cdot P_B[\underline{t}] - \underline{t} \cdot P_B[\underline{t}]$$

i.e. $\underline{t}$ is the lowest best response, contradiction. □

Proof of **Lemma 4.2**.

It is important to note that, from the follower's perspective, she only cares about the overall distribution of the leader's bids induced by his strategy: as long as the distribution of A's bids is unchanged, B's best response remain unchanged. Our idea is then to rearrange (sort) the leader's bids without changing the underlying distribution.

For any strategy s_A, we fix the bids' distribution D, i.e. the distribution of $s_A(v),\ v \sim F_1$, and rematch the leader's valuations to bids and create some new strategy $\breve{s}_A$. So the distribution of $\breve{s}_A(v),\ v \sim F_1$, is the same as distribution of $s_A(v),\ v \sim F_1$. In this process, the follower's best response remains unchanged, hence (2). In addition, her bid distribution is also fixed. As a result, for any single bid of A, the probability of winning is also the same. As a result, the overall winning probability is also unchanged.

Now look at the rank-and-bid based payment function, the leader's expected payment is still:

$$\int_{t \sim D} Pr[t, s_A] t dt,$$

unchanged after rematching.

To improve the leader's expected utility, which equals expected social welfare minus expected payment (fixed) we only need to increase the expected social welfare, given that the overall winning probability is fixed. Therefore, in rematching, we sort the strategy monotonically such that higher valuation is associated with higher winning probability, i.e., higher bid. In this way, we guarantee the total amount of fixed winning probability is allocated to the highest types, thus yielding the highest expected social welfare. From the reasoning above, we conclude that the leader's expected utility weakly increases, hence (3).

The above process can be thought of as rematching a bid that is in the top q quantile of the bid distribution to a type that is in the top q quantile of the type distribution, for all q. So there is no mixed strategy at any type, that is, $\breve{s}_A(v)$ is deterministic $\forall v$, hence (1). □

Proof of **Lemma 4.4**.

According to the definition of u:

$$u_B(y_1) = y_1 \cdot P_B[a] - a \cdot P_B[a] \geq y_1 \cdot P_B[b] - b \cdot P_B[b] \tag{4}$$

$$u_B(y_2) = y_2 \cdot P_B[b] - b \cdot P_B[b] \geq y_2 \cdot P_B[a] - a \cdot P_B[a] \tag{5}$$

$$\Rightarrow \quad (y_2 - y_1)[P_B[b] - P_B[a]] \geq 0 \qquad (4)+(5)$$

$$\Rightarrow \quad P_B[b] = P_B[a] \qquad (4) \text{ and } (5) \text{ are equalities.}$$

We substitute the last equality into the Equality (4), and get $aP_B[a] = bP_B[b]$. Because $a > b$, it must be $P_B[b] = P_B[a] = 0$. Furthermore $u_B(y_1) = u_B(y_2) = 0$.

Since we can set a to be any element in $S_B(y_1)$ as long as $a > b$, then $P_B[a] = 0$ for $\forall a \ s.t. \ b < a \in S_B(y_1)$. Since P_B is weakly increasing, then $P_B[a] = 0$ for all $a \in S_B(y_1)$. So the follower can achieve the highest utility 0 by bidding a, i.e. $a \in S_B(y_2)$. Hence $S_B(y_1) \subseteq S_B(y_2)$. □

Proof of **Lemma 4.5**.

We first prove $u_B(y)$ is weakly increasing. For any $y_1 < y_2$, if $u_B(y_1) = u_B(y_2) = 0$, the lemma is correct. Otherwise, pick $b_1 \in S_B(y_1)$, $b_2 \in S_B(y_2)$. According to Lemma 4.4, we have $b_1 \leq b_2$. By definition of u, we have

$$u_B(y_2) \geq (y_2 - b_1)P_B[b_1] \geq (y_1 - b_1)P_B[b_1] = u_B(y_1)$$

So, $u_B(y)$ is weakly increasing. Next we prove the continuity. For any $y_1 < y_2$, we have

$$\begin{aligned} & u_B(y_2) - u_B(y_1) \\ \leq \;& (y_2 - p^w(b_2))P_B[b_2] - (y_1 - b_2)P_B[b_2] \\ = \;& (y_2 - y_1)P_B[b_2] \\ \leq \;& y_2 - y_1 \end{aligned}$$

For $\forall \epsilon > 0$, as long as $y_2 \in (y_1 - \epsilon, y_1 + \epsilon)$, we have $|u_B(y_2) - u_B(y_1)| < \epsilon$. Thus, $u_B(y)$ is a continuous function. □

Proof of **Lemma 4.7**.

(1)

$$\frac{\partial eu}{\partial x}(y,x) = \frac{u_B(y)}{F_1^2[x]} f_1(x)$$

$\frac{\partial eu}{\partial x}(y,x)$ is continuous for both the arguments y and x when f_1 is continuous.

If $eu_B(y,x)$ has a breaking point y_0, then consider the definition $u_B(y) - F_1[x]y = -eu_B(y,x)F_1[x]$. The right hand side has a breaking point y_0, while the left hand side is continuous, contradiction. So $eu_B(y,x)$ is continuous in y.

(2) If $\exists x, y\ s.t.\ eu_B(x,y) > s_A(x)$, then consider the case when the follower with type y bids $s_A(x)$:

$$\begin{aligned} u_B(y) &\geq (y - s_A(x))F_1[x] \\ &> yF_1[x] - eu_B(y,x)F_1[x], \end{aligned}$$

which contradicts to the definition of $eu_B(x,y)$. □

Proof of **Lemma 4.9**.

(1) First we prove $(x, s_A^*(x))$ lies on some eu line. Suppose not, there exists number series $\{y_n\} \to y_0,\ s.t.\ s_A^*(x) = \lim_{n\to\infty} eu_B(y_n, x)$. Then

$$u_B(y_n) = F_1[x](y_n - eu_B(y_n, x))$$

We choose the limitation and get

$$u_B(y_0) = F_1[x](y_0 - s_A^*(x))$$

So $(x, s_A^*(x))$ lies on EU line $eu_B(y_0, \cdot)$.

(2) Since $eu_B(y, \cdot)$ weakly increases and $s_A^*(x) = \sup_{y\in[0,b_2]} eu_B(y,x)$, we have $s_A^*(x)$ weakly increases.

For any $x_1 < x_2$, s.t. $s_A^*(x_1) = s_A^*(x_2)$. Suppose $(x_1, s_A^*(x_1))$ lies on $eu_B(y_1, \cdot)$. Since $eu_B(y_1, \cdot)$ weakly increases, $eu_B(y_1, x_1) = eu_B(y_1, x_2) = s_A^*(x_1)$. Substitute it in Equation (1), we get $s_A^*(x_1) = y_1$, and $u_B(y_1) = 0$.

Furthermore, we get $eu_B(y_1, \cdot) = s_A^*(x_1) = s_A^*(x_2)$. Because s_A^* weakly increases, then $s_A^*(x) = s_A^*(x_1) \forall x \in (a_1, x_2]$, i.e., $s_A^*(x_1) = lim_{t\to a_1} s_A^*(t)$. Hence if $s_A^*(x) > \lim_{t\to a_1} s_A^*(t)$, $s_A^*(x)$ strictly increases.

(3) When $s_A^*(x) = \lim_{t\to a_1} s_A^*(t)$, $\exists x_1 < x_2 = x$ such that $s_A^*(x_1) = s_A^*(x_2)$. Then from (2.1), we know $s_A^*(x)$ lies on $eu_B(y_1, \cdot)$, where $y_1 = s_A^*(x_1)$. Moreover, $u_B(y_1) = F_1[x](y_1 - s_A^*(x_1)) = 0$.

(4) By Lemma 4.7, we know $eu_B(y,x) \leq s_A(x)$. We get $s_A^*(x) = \sup_y eu_B(y,x) \leq s_A(x)$ directly.

(5) At last we prove $s_A^*(x)$ is continuous. Suppose not, there is a breaking point x and a difference $d > 0$, s.t. $\forall \epsilon > 0$, $s_A^*(x+\epsilon) - s_A^*(x_2 - \epsilon) > d$.

Suppose $(x+\epsilon, s_A^*(x+\epsilon))$ lies on curve $eu_B(y(\epsilon), x)$. Then

$$
\begin{aligned}
(6) \qquad u_B(y(\epsilon)) &= y(\epsilon)F_1[x+\epsilon] - s_A^*(x+\epsilon)F_1[x+\epsilon] \\
u_B(y(\epsilon)) &= y(\epsilon)F_1[x-\epsilon] - eu_B(y(\epsilon), x-\epsilon)F_1[x-\epsilon] \\
(7) \qquad &> y(\epsilon)F_1[x-\epsilon] - (s_A^*(x+\epsilon) - d)F_1[x-\epsilon]
\end{aligned}
$$

(6)-(7), we get

$$
(8) \qquad y(\epsilon)(F_1[x+\epsilon] - F_1[x-\epsilon]) > [s_A^*(x+\epsilon) + s_A^*(x+\epsilon) - d]F_1[x-\epsilon]
$$

When $\epsilon \to 0$, the left hand side of (7) approaches zero, but the right hand side is strictly larger than zero. So $s_A^*(x)$ is continuous. □

Proof of **Lemma 4.10**.

(1) On one hand, because $s_A^*(x) \leq s_A(x)$, the follower's utility does not decrease no matter what the follower's value is. On the other hand, because $s_A^*(x) \geq eu_B(y,x)$, the follower's utility does not increase when the follower's value is y. Since the inequality holds for any y, the follower's utility does not increase no matter what the follower's value is. So the follower's utility is the same when the leader's strategy is $s_A(x)$ or $s_A^*(x)$.

(2) Pick $\forall t \in S_B(y)$, we have

$$
u_B(y) = (y-t)P_B[t] \leq (y-t)P_B^*[t] \leq u_B(y)
$$

The first inequality is due to $s \geq s_A^*$. The second inequality is true because the follower's utility is still $u_B(y)$ when the leader adopts s_A^*. Then these two inequalities are equalities. Hence $t \in S_B^*(y)$ and $S_B(y) \subseteq S_B^*(y)$.

Moreover, we have $(y-t)P_B[t] = (y-t)P_B^*[t]$. If $P_B[t] \neq P_B^*[t]$, it must be $y = t$. Since $u_B(y) \geq 0$, then we have $u_B(y) = 0$.

If $t > \lim_{x \to a_1} s_A^*(x)$, we can bid $\tilde{t}$ which is a little smaller than t, s.t. $P_B^*[\tilde{t}] > 0$, $\tilde{t} < y$, then the follower can achieve positive utility by bidding $\tilde{t}$, i.e. $u_B(y) > 0$, contradiction. Hence $t = \lim_{x \to a_1} s_A^*(x)$. □

Proof of **Lemma 4.11**.

(1)$P_B^*[s_A^*(x_0)] \geq F_1[x_0]$, the follower with value y_0 can achieve $u_B(y_0)$ by bidding $s_A^*(x_0)$. So $s_A^*(x_0) \in S_B^*(y_0)$.

(2)

$$\begin{aligned} u_B(y_0) &= (y_0 - s_A^*(x_0))P_B^*[s_A^*(x_0)] \\ &= (y_0 - s_A^*(x_0))F_1[x_0] \end{aligned}$$

So $eu_B(y_0, x_0) = s_A^*(x_0)$ by definition. □

Proof of **Theorem 4.14**.

For any A's value $x_0 > 0$, we consider the utility change between strategy s_A and strategy s_A^*. We prove the number of types, at which A's utility decreases, is countable, so the loss on these values is negligible and the total expected utility does not decrease.

(1) When $s_A^*(x_0) = lim_{t \to a_1} s_A^*(t)$ and $x_0 \neq \hat{x}$.

By Lemma 4.9, $(x_0, s_A^*(x_0))$ lies on $eu_B(y_0, \cdot)$, where $y_0 = s_A^*(x_0)$ and $u_B(y_0) = 0$.

By Lemma 4.13, we have

$$\begin{cases} s_A(x) = s_A^*(x) = s_A^*(\hat{x}) & x < \hat{x} \\ s_A(x) \geq s_A^*(x) > 0 & x > \hat{x} \end{cases}$$

(1.1) When the follower's type $y \leq y_0$.

We have $u_B(y) = 0$, $s_A^*(\hat{x}) > y$, and $P_B(s_A^*(\hat{x})) = P_B^*(s_A(\hat{x})) > 0$. If the follower bids $s_A^*(\hat{x})$, she will get a negative utility. So $s_A^*(\hat{x}) \notin S_B(y)$. Since $P_B[s_A^*(\hat{x})] = P_B^*[s_A^*(\hat{x})] = F_1[x] > 0$, by Lemma 4.4, $s_A^*(\hat{x}) > S_B^*(y)$, and $s_A^*(\hat{x}) > S_B(y)$. So the leader wins with value x_0 when the follower has value $y < y_0$ in both s_A and s_A^* strategies.

(1.2) When the follower's type $y > y_0$.

When using s_A^*, $\forall y > y_0$, the follower can just bid $s_A^*(x_0)$ and achieve a positive utility, so $u_B(y) > 0$. By Lemma 4.4, we have $S_B^*(y) \geq s_A^*(\hat{x})$. Then $S_B^*(y) \geq s_A^*(\hat{x})$. So the follower with value $y > y_0$ wins when the leader has value x_0 in both s_A and s_A^* strategies.

Combining (1.1) and (1.2), the leader's winning probability is always $F_2[y_0]$.

(2) When $s_A^*(x_0) \neq \lim_{t\to a_1} s_A^*(t)$

Let $eu_B(y_0, \cdot)$ be an equal utility curve that contains the point $(x_0, s_A^*(x_0))$. Then $s_A^*(x_0) \in S_B^*(y_0)$, by Lemma 4.11 Let $eu_B(y_1, \cdot)$ be an equal utility line that does not contain the point $(x_0, s_A^*(x_0))$.

(2.1) When $y_1 < y_0$,

(2.1.1) We want to prove when A adopts s_A^*, $B = y_1$ always loses against $A = x_0$. Otherwise, $\exists t \in S_B^*(y_1)$ $s.t.$ $s_A^*(x_0) \leq t$.

If $s_A^*(x_0) < t$, then $P_B^*[s_A^*(x_0)] = P_B^*[t]$ by Lemma 4.4. Since $s_A^*(x_0)$ strictly increases and $s_A^*(x_0) < t$, we have a contradiction.

If $s_A^*(x_0) = t$, then $s_A^*(x_0) \in S_B^*(y_1)$. By Lemma 4.11, we have $(x_0, s_A^*(x_0))$ lies on $eu_B(y_1, \cdot)$, contradiction.

(2.1.2) We want to prove when A adopts s_A, $B = y_1$ always loses against $A = x_0$. Otherwise, $\exists t \in S_B(y_1)$ $s.t.$ $t \geq s_A(x_0)$.

By Lemma 4.11, $s_A^*(x_0) \notin S_B^*(y_1)$(otherwise $(x_0, s_A^*(x_0))$ lies on $eu_B(y_1, \cdot)$). Then $s_A^*(x_0) \notin S_B(y_1)$ by Lemma 4.10. Thus $t \neq s_A^*(x_0)$. Since $t \geq s_A(x_0) \geq s_A^*(x_0)$, we have $t > s_A^*(x_0)$. Notice that $t \in S_B^*(y_1)$ and $s_A^*(x_0) \in S_B^*(y_0)$, we have $P_B^*[t] = P_B^*[s_A^*(x_0)]$ by Lemma 4.4. That contradicts to $t > s_A^*(x_0) > \lim_{t\to a_1} s_A^*(t)$.

Combining (2.1.1) and (2.1.2), the leader with x_0 beats follower with $y \leq y_0$ in both s_A^* and s_A.

(2.2) When $y_1 > y_0$.

(2.2.1) We want to prove when A adopts s_A^*, $B = y_1$ always beats $A = x_0$. By Lemma 4.11, we have $s_A^*(x_0) \in S_B^*(y_0)$ and $s_A^*(x_0) \notin S_B^*(y_1)$. By Lemma 4.10, we have $s_A^*(x_0) \notin S_B(y_1)$.

Since $y_1 > y_0$ and $S_B^*(y_0) \nsubseteq S_B^*(y_1)$, we have $S_B^*(y_1) \geq s_A^*(x_0)$ by Lemma 4.4.[6] So the follower with value y_1 beats the leader with value x_0 in strategy s_A^*.

(2.2.2) We want to prove when A adopts s_A, $B = y_1$ always beats $A = x_0$.

[6]Here we mean any element in set $S_B^*(y_1)$ is larger than $s_A^*(x_0)$.

Otherwise, $\exists t \in S_B(y_1)$ s.t. $t < s_A(x_0)$. Continue the proof above, since $S_B(y_1) \subseteq S_B^*(y_1)$, we have $S_B(y_1) > s_A^*(x_0)$. We then get

$$F_1[x_0] < P_B^*[t] = P_B[t] \leq F_1[x_0],$$

a contradiction. The first inequality is due to $t > s_A^*(x_0)$. The second equality follows from Lemma 4.10. The third inequality is due to $t < s_A(x_0)$.

Combining (2.2.1) and (2.2.2), when $y_1 > y_0$, the leader with x_0 loses to the follower with y_1 in both s_A^* and s_A.

(3) When $s_A^*(x_0) \neq \lim_{t\to a_1} s_A^*(t)$ and lies on a unique $eu_B(y_0,\cdot)$. Given part (2) above, the leader's winning probability is the same in both strategies s_A^* and s_A.

When $s_A^*(x_0) = \lim_{t\to a_1} s_A^*(t)$ and $x_0 \neq \sup\{x | s_A^*(x) = \lim_{t\to a_1} s_A^*(t)\}$. By part (1), the leader's winning probability is the same in both strategies s_A^* and s_A.

In these two cases, because of $s_A^* \leq s_A$ and the same winning probability, the expected utility of the leader weakly increases when changing from s_A to s_A^*.

In other cases, the leader's expected utility might decrease, but the loss on all these points is negligible.

Define V_1 as follows, we only need to prove $\#V_1$, the size of the set, is countable.

$$V_1 = \{(x, s_A^*(x)) | \text{point } p(x, s_A^*(x)) \text{lies on at least two eu lines, and } s_A^*(x) > \lim_{t\to a_1} s_A^*(t)\}$$

Pick arbitrary $x_0 \in V_1$, let $eu_B(y_1,\cdot)$ and $eu_B(y_2,\cdot)$, where $y_1 < y_2$, be two equal utility lines that pass point $(x_0, s_A^*(x_0))$. By Lemma 4.11, $s_A^*(x_0) \in S_B^*(y_1), S_B^*(y_2)$.

For any $y_3 > y_1$,

$$\begin{aligned} u_B(y_3) &\geq (y_3 - s_A^*(x_0))F_1[x_0] \\ &> (y_1 - s_A^*(x_0))F_1[x_0] = u_B(y_1) \end{aligned}$$

So $u_B(y_3) > 0$. Then consider $\forall y_3 \in Q s.t. y_1 < y_3 < y_2$, we have $u_B(y_2), u_B(y_3) > 0$.

By Lemma 4.4, we have

$$S_B(y_1) \leq S_B(y_3) \leq S_B(y_2)$$

So $S_B(y_3) = \{s_A^*(x_0)\}$, i.e., $s_A^*(x_0)$ is the unique best response of $B = y_3$. Now we can map any element in V_1 to a rational number. Since $s_A^*(x_0)$ is the unique best response of $B = y_3$, this mapping is injective. Thus $\#V_1$ is countable. Because the leader has a continuous distribution without point discontinuities, the loss on countable points is negligible. □

Proof of **Lemma 5.3**.

(1) If $Y(x)$ is not closed, then $\exists \{y_n\} \to y$ such that

$$u_B(y_n) = (y_n - s_A^*(x))F_1[x]$$

When n approaches infinity, we get

$$u_B(y) = (y - s_A^*(x))F_1[x]$$

Then $eu_B(y, \cdot)$ passes point $(x, s_A^*(x))$, so $y \in Y(x)$.

(2) If $\exists y_1 < \hat{y}$ such that $y_1 \in Y(x)$, then

$$\begin{aligned} 0 &= u_B(y_1) = (y_1 - s_A^*(x))F_1[x] \\ &< (\hat{y} - s_A^*(x))F_1[x] \leq u_B(\hat{y}) \end{aligned}$$

Which contradicts to $u_B(\hat{y}) = 0$.

(3) Suppose not, then $\exists y_1 \in Y(x_1), y_2 \in Y(x_2)$, such that $y_1 > y_2$. Now, we have

$$\begin{aligned} s_A^*(x_1) &\in S_B^*(y_1) \\ s_A^*(x_2) &\in S_B^*(y_2) \\ s_A^*(x_1) &< s_A^*(x_2) \end{aligned}$$

By Lemma 4.4, we have $P_B^*[s_A^*(x_1)] = P_B^*[s_A^*(x_2)] = 0$, which contradicts to $P_B^*[s_A^*(x_1)] \geq F_1[x_1]$.

(4) Pick any $y > \hat{y}$, let $t \in S_B^*(y)$. Obviously, $t \leq s_A^*(a_2)$. Since $u_B(y) > 0$, we have $P_B^*(t) > 0$, which leads to $t \geq \lim_x s_A^*(x)$.

(4.1) When $t > \lim_{x \to a_1} s_A^*(x)$.

Because s_A^* is continuous and $t \leq s_A^*(a_2)$, there exists x_0 such that $s_A^*(x_0) = t \in S_B^*(y)$. By Lemma 4.11, we have $(x_0, s_A^*(x_0))$ lies on $eu_B(y, \cdot)$.

(4.2) When $t = \lim_{x \to a_1} s_A^*(x)$.

Let $x_0 = F_1^{-1}[P_B^*[t]]$, then $s_A^*(x_0) = t$.

$$\begin{aligned} u_B(y) &= (y - t)P_B^*[t] \\ &= (y - s_A^*(x_0))F_1[x_0] \end{aligned}$$

So $(x_0, s_A^*(x_0))$ lies on $eu_B(y, \cdot)$.

Combining (4.1) and (4.2), we know $eu_B(y, \cdot)$ shares the common points with $s_A^*(x)$. So $\forall y > \hat{y}$, there exists x such that $s_A^*(x) \in S_B^*(y)$. Hence, $\cup_x Y(x)$ covers $(\hat{y}, b_2]$.

(5) If $Y(x)$ is not a unique number. Then there exists $y_1 < y_2 \in Y(x)$. For any $y \in (y_1, y_2)$.

$$Y(x_1) < y < Y(x_2) \ \forall x_1 < x < x_2$$

Because of the result in (4), it must be $y \in Y(x)$. So $Y(x)$ is an interval.

Since there are countable non-overlapping intervals, so for almost all x, $Y(x)$ contains only one element. □

Proof of **Lemma 5.5**.

We consider two cases.

(1) $s_A^*(x) = \lim_{t \to a_1} s_A^*(t)$

By Lemma 4.9, $(x, s_A^*(x))$ lies on $eu_B(y_0, \cdot)$, where $y_0 = p^w(s_A^*(x))$, $u_B(y_0) = 0$. Furthermore, $\forall y > y_0$, we have $u_B(y) > 0$ (by just bidding $s_A^*(x)$). So $\hat{y} = y_0$.

Because $\hat{y} \in Y(x)$ and $Y(x) \geq \hat{y}$, we have $g(x) = \min Y(x) = \hat{y}$. Then for $y \leq g(x)$, the follower bids zero. For $y > g(x)$, the follower bids larger than $s_A^*(x)$ (otherwise the winning probability is zero, which leads to zero utility). So the winning probability of the leader with value x is $F_2[g(x)]$.

(2) $s_A^*(x) > \lim_{t\to a_1} s_A^*(t)$.

By Lemma 4.11, if $(x, s_A^*(x))$ does not lie on $eu_B(y,\cdot)$, then $s_A^*(x) \notin S_B^*[y]$.

(2.1) When $g(x) > \hat{y}$

Since $u_B(g(x)) > 0$, then by Lemma 4.4, we have

$$S_B^*(y_1) \leq S_B^*(g(x)) \leq S_B^*(y_2) \; y_1 < g(x) < y_2$$

Then $S_B^*(y_1) < s_A^*(x) \leq S_B^*(y_2)$. Hence, bidding $s_A^*(x)$, the leader's winning probability is $F_2[g(x)]$.

(2.2) When $g(x) = \hat{y}$

By definition of $\hat{y}$, we have $u_B(y) > 0$ for all $y > \hat{y}$. By Lemma 4.4, we have $g(x) \leq S_B^*(y)$. For $y > g(x)$, the follower bids greater than $s_A^*(x)$. For $y \leq g(x)$, the follower bids zero. So the winning probability of the leader with value x is $F_2[g(x)]$. □

Proof of **Lemma 5.6**.

(1) By Lemma 5.3(3), we know $g(x)$ weakly increases.

(2) Suppose not, then there exists x_0 and d such that $\forall x < x_0$, $g(x) < g(x_0) - d$. By Lemma 5.3(4), pick any $y \in (g(x_0) - d, g(x_0))$, there exists x_1 such that $y \in Y(x_1)$. Then $x_1 < x_0$, we also have:

$$g(\frac{x_1 + x_0}{2}) \geq \sup Y(x_1) \geq y > g(x_0) - d,$$

which contradicts the supposition. □

Proof of **Theorem 5.7**.

(1) For any x_0, we prove $s_A^*(x)$ is right-hand differentiable at point x_0. The proof for the left-hand case is similar.

If $x_2 > x_0$ is close enough to x_0, by Lemma 5.3, we know $\underline{Y}(x_2)$ is close enough to $\overline{Y}(x_0)$. Denote $y_0 = \overline{Y}(x_0) = g(x_0)$ and $y_2 = \underline{Y}(x_2)$, where $\underline{Y} = \inf Y, \overline{Y} = \sup Y$.

Since $(x_0, s_A^*(x_0))$ lies on $eu_B(y_0, \cdot)$ and $(x_2, s_A^*(x_2))$ lies on $eu_B(y_2, \cdot)$, we have

$$
\begin{aligned}
(9) \qquad s_A^*(x_0) &= eu_B(y_0, x_0) = y_0 - \frac{u_B(y_0)}{F_1[x_0]} \\
(10) \qquad s_A^*(x_0) &\geq eu_B(y_2, x_0) = y_2 - \frac{u_B(y_2)}{F_1[x_0]} \\
(11) \qquad s_A^*(x_2) &= eu_B(y_2, x_2) = y_2 - \frac{u_B(y_2)}{F_1[x_2]} \\
(12) \qquad s_A^*(x_2) &\geq eu_B(y_0, x_2) = y_0 - \frac{u_B(y_0)}{F_1[x_2]}
\end{aligned}
$$

Taking the difference between Equations (11) and (10) and divide it by $x_2 - x_0$, we get

$$
(13) \qquad \frac{s_A^*(x_2) - s_A^*(x_0)}{x_2 - x_0} \leq \frac{u_B(y_2)(\frac{1}{F_1[x_0]} - \frac{1}{F_1[x_2]})}{x_2 - x_0}
$$

Since u and F_1 are continuous, we have

$$
\lim_{x_2 \to x_0} \frac{s_A^*(x_2) - s_A^*(x_0)}{x_2 - x_0} \leq \frac{u_B(y_0)}{F_1^2[x_0]} f_1(x_0)
$$

Similarly, if we take the difference between Equations (12) and (9), we can get a lower bound same as the upper bound. So $s_A^*(x)$ is right-hand differentiable at point x_0, and

$$
(s_A^*)'(x) = \frac{u_B(\overline{Y(x)})}{F_1^2[x]} f_1(x).
$$

(2) Following the above results, we have

$$
\begin{aligned}
\partial_+ s_A^*(x) &= \frac{f_1(x) F_1[x](\overline{Y}(x) - s_A^*(x))}{F_1^2[x]} \\
&= \frac{f_1(x)\overline{Y}(x) - s_A^*(x) f_1(x)}{F_1[x]} \\
f_1(x)\overline{Y}(x) &= \partial_+ s_A^*(x) \cdot F_1[x] + s_A^*(x) f_1(x) \\
f_1(x)\overline{Y}(x) &= \partial_+ [s_A^*(x) F_1[x]]
\end{aligned}
$$

Similarly, we have

$$
f_1(x) g(x) = \partial_- [s_A^*(x) F_1[x]]
$$

Because $\overline{Y}(x) = g(x)$ for almost all x, then when we do integration on both right and left derivatives, the difference vanishes, that is,

$$\int_{a_1}^{x} f_1(t)g(t)dt = s_A^*(x)F_1[x].$$

□

Proof of **Theorem 5.9**.

For any $g \in O_2$, define $s_A = M_2(g)$ to be the solution of

$$\int_{a_1}^{x} f_1(t)g(t)dt = s_A(x)F_1[x]$$

(1) We want to show M_2 is a mapping from $g \in O_2$ to $s_A \in O_1$. We need to prove that $s_A(x)$ does not change after sorting and smoothing, i.e. $M_2(g) \in O_1$.

$$\begin{aligned} s'_A(x) &= -\frac{f_1(x)}{F_1^2[x]}\int_{a_1}^{x} f_1(t)g(t)dt + \frac{f_1(x)g(x)}{F_1[x]} \\ &= \frac{f_1(x)}{F_1[x]}[g(x) - \frac{\int_{a_1}^{x} f_1(t)g(t)dt}{F_1[x]}] \end{aligned}$$

Since g weakly increases, s_A weakly increases.

First, we define $u_B(y,t)$ and $\tilde{u}(y,t)$ as follows:

$$\begin{aligned} u_B(y,t) &= [y - s_A(t)]P_B[t] \\ \tilde{u}(y,t) &= [y - s_A(t)]F_1[t] \end{aligned}$$

So $u_B(y,t)$ is the follower's utility with type y and bidding strategy $s_A(t)$. Since s_A weakly increases, we have $P_B[t] > F_1[t]$, then $u_B(y,t) \geq \tilde{u}(y,t)$. Furthermore $\max_t u_B(y,t) \geq \max_t \tilde{u}(y,t)$. For any t, $u_B(y,t) = \tilde{u}(y, F_1^{-1}[P_B^*[s_A(t)]])$, that is, $\tilde{u}(y,t)$ can achieve any value that $u_B(y,t)$ achieves. Thus we have $u_B(y) = \max_t u_B(y,t) = \max_t \tilde{u}(y,t)$, to compute $u_B(y)$, we only need to focus on $\tilde{u}(y,t)$ instead.

Next, we prove that $(x, s_A(x))$ lies on $eu_B(g(x),\cdot)$. Consider the utility of the follower with value $g(x)$. Since the leader's lowest bid might be zero, the follower bids between the highest and the lowest of the leader's bid. Then the derivative $\frac{\partial \tilde{u}}{\partial t}$ should be zero.

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t}(g(x),t) &= [g(x)-s_A^*(t)]f_1(t)-(s_A^*)'(t)\cdot F_1[t]\\ &= g(x)f_1(t)-g(t)f_1(t)\end{aligned}$$

It is easy to see that $\max_t \tilde{u}(g(x),t)=\tilde{u}(g(x),x)$. Then $u_B(g(x))=\max_t \tilde{u}(y,t)=\tilde{u}(g(x),x)=[g(x)-s_A(x)]F_1[t]$. By definition, the point $(x,s_A(x))$ lies on $eu_B(g(x),\cdot)$.

Let s_A^* be defined as before based on s_A. Then $s_A(x)=eu_B(g(x),x)\leq s_A^*(x)\leq s_A(x)$, it must be $s_A(x)=s_A^*(x)$. That means $s_A(x)$ does not change after sorting and smoothing, so $M_2(s_A)\in O_1$.

(2) We now prove that $M_1\circ M_2=I$. Suppose otherwise, then there exists g such that $M_1(M_2(g))=\tilde{g}\neq g$. Let $s_A=M_2(g)$. By Lemma 5.7, we have $\int_{a_1}^x f_1(t)\tilde{g}(t)dt=s_A^*(x)F_1[x]$. On the other side, by the method used in M_2, we have $\int_{a_1}^x f_1(t)g(t)dt=s_A^*(x)F_1[x]$. So $\int_{a_1}^x f_1(t)\tilde{g}(t)dt=\int_{a_1}^x f_1(t)g(t)dt$ for any x. Since g and $\tilde{g}$ are left continuous, so if there exists $g\neq\tilde{g}$ for some number x_0 then $g\neq\tilde{g}$ for some interval on the left side of x_0. Hence, the equation above will not always hold on that interval. Contradiction.

By (2), M_1 is a surjection. Obviously, M_1 is injective. So M_1 is a bijective mapping between O_1 and O_2. □

Proof of **Theorem 6.1**.

(1) We consider the first case, and the other case is similar. Suppose we set

$$j(x)=\begin{cases}0 & x\notin L\\ \lim_{t\to(\sup L)^+} g(t)-g(x) & x\in L\end{cases}$$

then Equation (2) becomes positive which contradicts with g's optimality.

(2) Create function $\tilde{g}$ that has no image on $(0,b_1)$.

$$\tilde{g}(x)=\begin{cases}0 & g(x)\in(0,b_1)\\ g(x) & o.w.\end{cases}$$

$\tilde{g} \leq g(x)$, then s_A^* based on $\tilde{g}$ is smaller than s_A^* based on g. While they keep the same winning probability as long as the winning probability $F_2[g]$ is non-zero. So the expected utility weakly increases when using $\tilde{g}$ instead of g. □

Proof of **Lemma 6.2**. Let $t_0 = \sup\{t|g(t) = 0\}$. If $g(t) > 0, \forall t$, let $t_0 = a_1$. When F_2 is uniformly distributed, $f_2(g(x))$ does not change on $(t_0, a_2]$, we denote it f_2. It's easy to see that $h(x)$ is continuous on $(t_0, a_2]$.

(1) When $t > t_0$, by Theorem 6.1, $g(t) \geq b_1$, $f_2(g(t))$ is a constant.

$$\begin{aligned} h(x) &= f_1(x)f_2 \cdot (x - \frac{1}{F_1[x]}\int_{t_0}^{x} g(t)f_1(t)dt + \int_{x}^{a_2} \frac{-f_1(t)}{F_1[t]}(g(t) - b_1)dt) \\ (\frac{h(x)}{f_1(x)})'\frac{F_1^2(x)}{f_1(x)f_2} &= \frac{F_1^2(x)}{f_1(x)} + \int_{t_0}^{x} g(t)f_1(t)dt - b_1F_1[x] \quad (14) \\ ((\frac{h(x)}{f_1(x)})'\frac{F_1^2(x)}{f_1(x)f_2})' &= \frac{2F_1[x]f_1^2(x) - F_1^2(x)f_1'(x)}{f_1^2(x)} + g(x)f_1(x) - b_1f_1(x) \end{aligned}$$

Since $\forall t, 2f_1^2(t) - F_1[t]f_1'(t) \geq 0$, then $\left((\frac{h(x)}{f_1(x)})'\frac{F_1^2(x)}{f_1(x)f_2}\right)' > 0$, then there is at most one solution for $(\frac{h(x)}{f_1(x)})' = 0$, then there are at most two crosses for $\frac{h(x)}{f_1(x)} = 0$, i.e. $h(x) = 0$. Assume the two crosses are t_1 and t_2. There are several cases for the sign of h to change. In any of these cases, we can prove that function g consists of at most three values. For example, let us consider the following case:

$$h(x) = \begin{cases} > 0 & x \in (t_0, t_1) \\ < 0 & x \in (t_1, t_2) \\ > 0 & x \in (t_2, a_2) \end{cases}$$

By Theorem 6.1 $g(x) = g(t_1), x \in (t_0, t_2]$ is a constant and $g(x) = b_2, x \in (t_2, a_2]$ is a constant. So there are at most two values in $g(x)$ in $(t_0, a_2]$. Counting the part where g is zero, there are at most three values.

(2) Since $b_1 = 0$, $f_2(0) = f_2$, we have that h is continuous on the whole range. When $x \leq t_0$,

$$\frac{h(x)}{f_1(x)} = f_2x + \int_{t_0}^{a_2} \frac{-f_1(t)}{F_1[t]}g(t)f_2dt$$

So $(\frac{h(x)}{f_1(x)})' > 0, \ \forall x \in (a_1, t_0]$. Moreover, $h(x)$ is continuous at point t_0. Combined with Equation (14), we have $\frac{h(x)}{f_1(x)}$ strictly increases on the whole range $[a_1, a_2]$. Then equation $h(x) = 0$ has at most one solution. When there is no solution, since $\lim_{t \to a_2} h(t) > 0$, it must be $h(x) > 0$ for the whole range. It also means that $t_0 = a_2 \int_{t_0}^{a_2} \frac{f_1(t)}{F_1[t]} dt$ has no solution. If $g(x)$ is optimal, then it must be $g(x) = b_2, \forall x \in [a_1, a_2]$.

So we only need to consider the nontrivial case that there is one solution for $h(x) = 0$. Assume

$$h(x) = \begin{cases} < 0 & x \in (a_1, t_0) \\ > 0 & x \in (t_0, a_2) \end{cases}$$

By Theorem 6.1, we have $g(x) = 0, x \in [a_1, t_0)$ and $g(x) = b_2, x \in (t_0, b_2]$. Since $g(x)$ is left continuous, it should be $g(t_0) = 0$.

Thus the form of $g(x)$ is fixed, then we compute the optimal breaking point t_0, from $h(t_0) = 0$, we have

$$t_0 = b_2 \int_{t_0}^{a_2} \frac{f_1(t)}{F_1[t]} dt.$$

In conclusion, when $t_0 = a_2 \int_{t_0}^{a_2} \frac{f_1(t)}{F_1[t]} dt$ has a solution,

$$g(x) = \begin{cases} 0 & x \in (a_1, t_0) \\ b_2 & x \in (t_0, a_2) \end{cases}, \quad \text{where } t_0 = a_2 \int_{t_0}^{a_2} \frac{f_1(t)}{F_1[t]} dt.$$

When there is no solution, $g(x) = b_2, \forall x \in [a_1, a_2]$. □

Proof of **Lemma 6.3**.

By Theorem 6.2, function

$$t_0 = 1 \int_{t_0}^{1} \frac{1}{x} dx$$

has a solution $t_0 \approx 0.567$. When $x > t_0$, we have $s_A^*(x) = \frac{1}{F_1[x]} \int_{t_0}^{x} f_1(t) g(t) dt = 1 - \frac{t_0}{x}$ The expected utility is

$$\int_{t_0}^{1} (x - 1 + \frac{t_0}{x} dx \approx 0.228$$

□

B. Proofs for the rank-and-bid based auctions

We provide the proof for Lemma 7.4, Theorem 5.9, and Thereom 7.6. The proofs for other theorems are similar.

Proof of **Lemma 7.4**.

First, notice that $p^w(t)a + p^p(t)$ strictly increases, so solution t exists.

Second, $t(a, b)$ is continuous. Suppose not, let $t(a, b)$ jumps at point (a, b). Then $ap^w(t) + p^p(t) + b$ jumps at (a, b), it could not always be zero, leading to contradiction.

Third, $t(a, b)$ is differentiable. Suppose otherwise, $t(a, b)$ does not have partial derivatives with respect to a, at point (a_0, b_0). Then

$$\begin{aligned} & \exists k_1 > k_2, \{\tilde{a}_i\}, \{a_i\} \to a_0 \\ s.t. \quad & t(\tilde{a}_i, b_0) \geq t(a_0, b_0) + k_1(\tilde{a}_i - a_0) \\ & t(a_i, b_0) \leq t(a_0, b_0) + k_2(a_i - a_0) \end{aligned}$$

$$\begin{aligned} (15) &\geq p^w(t(a, b_0) + k_1(\tilde{a}_i - a))\tilde{a}_i + p^p(t(a) + k_1(\tilde{a}_i - a_0)) + b_0 \\ (16) &= p^w(t(a_0, b_0))a_0 + p^p(t(a_0, b_0)) + b_0 \\ 0 &\geq p^w(t(x) + k_1(x_i - x))F_1[x_i] + p^p(t(x) + k_1(x_i - x)) \qquad (15) - (16) \\ 0 &\geq p^w(t(a, b_0) + k_1(\tilde{a}_i - a))\tilde{a}_i - p^w(t(a_0, b_0))a_0 \\ & \quad p^p(t(a) + k_1(\tilde{a}_i - a_0)) - p^p(t(a_0, b_0)) \end{aligned}$$

Divide $a_i - a_0$ on both sides and consider the limitation when i approaches infinity, we have

$$0 \geq (p^w)'(t(a_0, b_0))k_1 a_0 + p^w(t(a_0, b_0)) + (p^p)'(t(a_0))k_1$$

Similarly, we have

$$0 \leq (p^w)'(t(a_0, b_0))k_2 a_0 + p^w(t(a_0, b_0)) + (p^p)'(t(a_0))k_2$$

These two equations together contradicts the fact that $k_1 > k_2$ and $(p^w)' + (p^p)' > 0$. So $t(a, b)$ is differentiable.

We differentiate $b + ap^w(t) + p^p(t) = 0$ on a and b, and have

$$\begin{aligned}(p^p)'(t)t'_a + (p^w)'(t)t'_a a + p^w(t) = 0\\ 1 + (p^w)'(t)t'_b a + p^w(t)t'_b = 0\end{aligned}$$

Since t,$(p^w)'$ and $(p^p)'$ is continuous, it's easy to see these two derivatives are continuous. □

Proof of **Theorem 5.9**.

For any $g \in O_2$, define $s_A = M_2(g)$ to be the solution of

$$\int_{a_1}^{x} f_1(t)g(t)dt = p^w(s_A(x))F_1[x] + p^p(s_A(x)) \tag{17}$$

(1) We want to prove M_2 is a mapping from $g \in O_2$ to $s_A \in O_1$. We need to prove that $s_A(x)$ does not change after sorting and smoothing, i.e. $M_2(g) \in O_1$.

Let $a = F_1[x]$, $b = -\int_{a_1}^{x} f_1(t)g(t)dt$, $s_A(x) = t(a(x), b(x))$. By Lemma 7.4, $s_A(x)$ is unique and differentiable. Since g weakly increases, by Equation (17), $g(x)F_1[x] \geq p^w(s_A(x))F_1[x]$. Furthermore, $g(x) \geq p^w(s_A(x))$.

$$\begin{aligned} s'_A(x) &= \frac{\partial t}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial t}{\partial b}\cdot\frac{\partial b}{\partial x}\\ &= \frac{f_1(x)g(x) - p^w(s_A(x))f_1(x)}{(p^w)'(s_A(x))F_1[x] + (p^p)'(s_A(x))} \geq 0\end{aligned}$$

So s_A weakly increases.

We define $u_B(y,t)$ and $\tilde{u}(y,t)$ as follows:

$$\begin{aligned} u_B(y,t) &= [y - p^w(s_A(t))]P_B[t] - p^p(s_A(t))\\ \tilde{u}(y,t) &= [y - p^w(s_A(t))]F_1[t] - p^p(s_A(t))\end{aligned}$$

So $u_B(y,t)$ is the follower's utility with type y and bidding $s_A(t)$, Since s_A weakly increases, we have $P_B[t] > F_1[t]$, then $u_B(y,t) \geq \tilde{u}(y,t)$. Furthermore $\max_t u_B(y,t) \geq \max_t \tilde{u}(y,t)$. For any t, $u_B(y,t) = \tilde{u}(y, F_1^{-1}[P_B^*[s_A(t)]])$, i.e. $\tilde{u}(y,t)$ can achieve any value that $u_B(y,t)$ achieves. Thus we have $u_B(y) = \max_t u_B(y,t) = \max_t \tilde{u}(y,t)$, to compute $u_B(y)$, we only need to focus on $\tilde{u}(y,t)$ instead.

Next, we prove that $(x, s_A(x))$ lies on $eu_B(g(x),\cdot)$. Consider utility of the follower with value $g(x)$. Since, leader's lowest bid is zero ($s_A(a_1) = 0$), the follower bids between the highest and the lowest of the leader's bid. Then the derivative $\frac{\partial \tilde{u}}{\partial t}$ should be zero.

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t}(g(x), t) &= [g(x) - p^w(s_A^*(t))]f_1(t) - (p^w)'(s_A^*(t)) \cdot (s_A^*)'(t) \cdot F_1[t] \\ &\quad -(p^p)'(s_A^*(t)) \cdot (s_A^*)'(t) \\ &= g(x)f_1(t) - g(t)f_1(t)\end{aligned}$$

It's easy to see that $\max_t \tilde{u}(g(x), t) = \tilde{u}(g(x), x)$. Then $u_B(g(x)) = \max_t \tilde{u}(y, t) = \tilde{u}(g(x), x) = [g(x) - p^w(s_A(x))]F_1[t] - p^p(s_A(x))$. By definition, the point $(x, s_A(x))$ lies on $eu_B(g(x), \cdot)$.

Let s_A^* is defined as before based on s_A. Then $s_A(x) = eu_B(g(x), x) \leq s_A^*(x) \leq s_A(x)$, it must be $s_A(x) = s_A^*(x)$. That means $s_A(x)$ does not change after sorting and smoothing, so $M_2(s_A) \in O_1$.

(2) We now prove that $M_1 \circ M_2 = I$. Suppose otherwise, then there exists g such that $M_1(M_2(g)) = \tilde{g} \neq g$. Let $s_A = M_2(g)$. By Lemma 5.7, we have $\int_{a_1}^x f_1(t)\tilde{g}(t)dt = p^w(s_A^*(x))F_1[x] + p^p(s_A^*(x))$. On the other hand, by the method used in M_2, we have $\int_{a_1}^x f_1(t)g(t)dt = p^w(s_A^*(x))F_1[x] + p^p(s_A^*(x))$. So $\int_{a_1}^x f_1(t)\tilde{g}(t)dt = \int_{a_1}^x f_1(t)g(t)dt$ for any x. Since g and $\tilde{g}$ are left continuous, if there exists $g \neq \tilde{g}$ for some number x_0 then $g \neq \tilde{g}$ for some interval on the left side of x_0. Hence, the equation above will not always hold on that interval. Contradiction.

By (2), M_1 is a surjection. Obviously, M_1 is injective. So M_1 is a bijective mapping between O_1 and O_2. □

Proof of **Theorem 7.6**. (1) For any x_0, we prove $s_A^*(x)$ is right-hand differentiable at point x_0. The proof for the left-hand case is similar.

If $x_2 > x_0$ is close enough to x_0, by Lemma 5.3, we know $\underline{Y}(x_2)$ is close enough to $\overline{Y}(x_0)$. Denote $y_0 = \overline{Y}(x_0) = g(x_0)$ and $y_2 = \underline{Y}(x_2)$ ($\underline{Y} = \inf Y, \overline{Y} = \sup Y$).

Let $Q(a, b)$ denote the solution t of

$$b + ap^w(t) + p^p(t) = 0$$

By Lemma 7.4, $Q(a,b)$ exists and

$$\frac{\partial Q}{\partial a}(a,b) = \frac{-p^w(Q)}{a(p^w)'(Q)+(p^p)'(Q)} \qquad \frac{\partial Q}{\partial b}(a,b) = \frac{-1}{a(p^w)'(Q)+(p^p)'(Q)}$$

We have $\frac{\partial Q}{\partial a}$ and $\frac{\partial Q}{\partial b}$ are continuous. We should also keep in mind that $\frac{\partial Q}{\partial a}$ and $\frac{\partial Q}{\partial b}$ are negative.

Since $(x_0, s_A^*(x_0))$ lies on $eu_B(y_0,\cdot)$ and $(x_2, s_A^*(x_2))$ lies on $eu_B(y_2,\cdot)$, we have

$$\begin{aligned} s_A^*(x_0) &= eu_B(y_0,x_0) \qquad s_A^*(x_0) \geq eu_B(y_2,x_0) \\ s_A^*(x_2) &= eu_B(y_2,x_2) \qquad s_A^*(x_2) \geq eu_B(y_0,x_2) \end{aligned}$$

Rewrite these equations in terms of Q we have

$$s_A^*(x_0) = Q(F_1[x_0], u_B(y_0) - y_0 F_1[x_0]) \tag{18}$$

$$s_A^*(x_0) \geq Q(F_1[x_0], u_B(y_2) - y_2 F_1[x_0]) \tag{19}$$

$$s_A^*(x_2) = Q(F_1[x_2], u_B(y_2) - y_2 F_1[x_2]) \tag{20}$$

$$s_A^*(x_2) \geq Q(F_1[x_2], u_B(y_0) - y_0 F_1[x_x]) \tag{21}$$

Equation (20)-(19) and divide it by $x_2 - x_0$, we get

$$\frac{s_A^*(x_2) - s_A^*(x_0)}{x_2 - x_0} \leq \frac{Q(F_1[x_2], u_B(y_2) - y_2F_1[x_2]) - Q(F_1[x_0], u_B(y_2) - y_2F_1[x_0])}{x_2 - x_0} \tag{22}$$

Since u and F_1 are continuous, we have $\lim_{x_2\to x_0} u_B(y_2) - y_2F_1[x_2] = \lim_{x_2\to x_0} u_B(y_2) - y_2F_1[x_0] = u_B(y_0) - y_0F_1[x_0]$.

Since $\frac{\partial Q}{\partial a}$ and $\frac{\partial Q}{\partial b}$ are continuous, for any $\epsilon > 0$, we can pick x_2 small enough such that when $(a,b) \in [F_1[x_0], F_1[x_2]] \times [u_B(y_2) - y_2F_1[x_2], u_B(y_2) - y_2F_1[x_0]]$ we have

$$\begin{aligned} \left(\frac{\partial Q}{\partial a}(a,b), \frac{\partial Q}{\partial b}(a,b)\right) \in\ & \left[\frac{\partial Q}{\partial a}(F_1[x_0], u_B(y_0) - y_0F_1[x_0]) \cdot (1 \pm \epsilon)\right] \\ & \times \left[\frac{\partial Q}{\partial b}(F_1[x_0], u_B(y_0) - y_0F_1[x_0]) \cdot (1 \pm \epsilon)\right] \end{aligned}$$

We use integration of the derivatives to represent the numerator of Equation (22). Since the derivatives are very close, we can have a good upper bound.

$$\begin{aligned}
& Q(F_1[x_2], u_B(y_2) - y_2F_1[x_2]) - Q(F_1[x_0], u_B(y_2) - y_2F_1[x_0]) \\
= & \int_{F_1[x_0]}^{F_1[x_2]} \frac{\partial Q}{\partial a}(t, u_B(y_2) - y_2F_1[x_2])dt - \int_{u_B(y_2)-y_2F_1[x_2]]}^{u_B(y_2)-y_2F_1[x_0]} \frac{\partial Q}{\partial b}(F_1[x_0], t)dt \\
\leq & (F_1[x_2] - F_1[x_0])(1-\epsilon)\frac{\partial Q}{\partial a}(F_1[x_0], u_B(y_0) - y_0F_1[x_0]) \\
& -y_2(F_1[x_2] - F_1[x_0])(1+\epsilon)\frac{\partial Q}{\partial b}(F_1[x_0], u_B(y_0) - y_0F_1[x_0])
\end{aligned}$$

The inequality is due to the fact that $\frac{\partial Q}{\partial a}$ and $\frac{\partial Q}{\partial b}$ are negative. Then the limitation of Equation (22) would be

$$\begin{aligned}
\lim \frac{s_A^*(x_2) - s_A^*(x_0)}{x_2 - x_0} \leq & f_1(x_0)(1-\epsilon)\frac{\partial Q}{\partial a}(F_1[x_0], u_B(y_0) - y_0F_1[x_0]) \\
& -y_2 f_1(x_0)(1+\epsilon)\frac{\partial Q}{\partial b}(F_1[x_0], u_B(y_0) - y_0F_1[x_0])
\end{aligned}$$

Since it works for any ϵ, we have

$$\begin{aligned}
\lim \frac{s_A^*(x_2) - s_A^*(x_0)}{x_2 - x_0} \leq & f_1(x_0)\frac{\partial Q}{\partial a}(F_1[x_0], u_B(y_0) - y_0F_1[x_0]) \\
& -y_2 f_1(x_0)\frac{\partial Q}{\partial b}(F_1[x_0], u_B(y_0) - y_0F_1[x_0]) \\
= & \frac{-f_1(x_0)p^w(s_A^*(x_0)) + f_1(x_0)g(x_0)}{F_1[x_0](p^w)'(s_A^*(x_0)) + (p^p)'(s_A^*(x_0))} \\
= & \frac{\partial eu}{\partial x}(\overline{Y}(x), x)
\end{aligned}$$

Similarly, if we use Equation (21)-(18), we can get a lower bound the same way as we get the upper bound. So $s_A^*(x)$ is right-hand differentiable at point x_0, and

$$(s_A^*)'(x) = \frac{\partial eu}{\partial x}(\overline{Y}(x), x)$$

(2) Following the above argument, we have

$$\begin{aligned}
\partial_+ s_A^*(x) &= \frac{f_1(x)\overline{Y}(x) - p^w(s_A^*(x))f_1(x)}{(p^w)'(s_A^*(x))F_1[x] + (p^p)'(s_A^*(x))} \\
f_1(x)\overline{Y}(x) &= \partial_+ s_A^*(x) \cdot (p^w)'(s_A^*(x))F_1[x] + p^w(s_A^*(x))f_1(x) \\
& \quad +\partial_+ s_A^*(x) \cdot (p^p)'(s_A^*(x)) \\
f_1(x)\overline{Y}(x) &= \partial_+[p^w(s_A^*(x))F_1[x] + p^p(s_A^*(x))]
\end{aligned}$$

Similarly, we have

$$f_1(x)g(x) = \partial_-[p^w(s_A^*(x))F_1[x] + p^p(s_A^*(x))]$$

Because $\overline{Y}(x) = g(x)$ for almost all x, then when we do integration on both right and left derivatives, the difference vanishes, that is,

$$\int_{a_1}^{x} f_1(t)g(t)dt = p^w(s_A^*(x))F_1[x] + p^p(s_A^*(x)) - p^p(s_A^*(a_1)).$$

□

Proof of **Theorem 7.8**.

By Theorem 7.6, we get

$$s_A^*(x) = \int_{a_1}^{x} f_1(t)g(t)dt + s_A^*(a_1)$$

The leader's expected utility is

$$\int_{a_1}^{a_2} \{xF_2[g(x)] - s_A^*(x)\}f_1(x)dx$$

When the function g is fixed, if $s_A^*(a_1)$ becomes smaller, $s_A^*(x)$ becomes smaller, then we get higher expected utility. So in the optimal strategy, we should have $s_A^*(a_1) = 0$, and $s_A^*(x) = \int_{a_1}^{x} f_1(t)g(t)dt$. □

Proof of **Theorem 7.9**.

Let $t_0 = \sup\{t|g(t) = 0\}$. If t_0 does not exist, let $t_0 = a_1$. We first consider the case when $x \in (t_0, a_2]$. When f_2 is weakly increasing,

$$\frac{h(x)}{f_1(x)} = xf_2(g(x)) - 1 + F_1[x]$$

strictly increases. Then there is at most one cross for $\frac{h(x)}{f_1(x)} = 0$, i.e. $h(x) = 0$. If h keeps positive, then $g(x)$ should be a constant b_2. If h keeps negative, then $g(x)$ should be constantly 0. Assume the cross is at t_1.

$$h(x) = \begin{cases} < 0 & x \in (t_0, t_1) \\ > 0 & x \in (t_1, a_2) \end{cases}$$

Then $g(x), x \in [t_0, t_1)$ is constantly zero and $g(x), x \in [t_1, a_2]$ is a constant b_2. Otherwise we can decrease $g(x), x \in (t_0, t_1)$ and increase $g(x), x \in (t_1, a_2]$, which contradicts with the optimality of g.

Consider the case when $x \in [a_1, t_0]$. We have $t_0 = t_1$. So there are at most two values in $g(x)$ in $[a_1, a_2]$. Now we compute the optimal t_0. Since the case for g has only one value is a degenerated case of the two values, we only need to consider the two-values case. We first compute s^*_A,

$$s^*_A(x) = \int_{a_1}^{x} f_1(t)g(t)dt = \begin{cases} 0 & x \in [a_1, t_0) \\ (F_1[x] - F_1[t_0]) & x \in [t_0, a_2] \end{cases}$$

Then the leader's expected utility is

$$\begin{aligned} u_A(s^*_A) &= \int_{a_1}^{a_2} f_1(x)[xF_2[g(x)] - s^*_A(x)]dx \\ &= \int_{t_0}^{a_2} f_1(x)[x - F_1[x]b_2 + F_1[t_0]b_2]dx \\ &= \int_{t_0}^{a_2} f_1(x)[x - F_1[x]b_2]dx + F_1[t_0]b_2[1 - F_1[t_0]] \end{aligned}$$

The derivative of the utility with respect to t_0 is

$$f_1(t_0)[b_2 - t_0 - b_2F_1[t_0]]$$

Then optimal t_0 must be the solution of $b_2 - t - b_2F_1[t] = 0$, and it's unique. □

C. The assumptions are not necessary

The idea to relax the assumptions is that we can achieve the same largest utility with or without the assumptions and we prove that the optimal strategy in different settings are similar.

C.1. Assumption 2.1 is not necessary

Without Assumption 2.1, the follower can give any best response. We prove that the maximal expected utility of the leader under both Assumptions is the same as under only Assumption 2.2.

First, the smoothing method does not depend on Assumption 2.1. The proof of Theorem 4.14 does not depend on Assumption 2.1. From the same s_A, we can create the same s^*_A no matter whether Assumption 2.1 is imposed.

Second, from the proof of Theorem 4.14, we know there are only countable points $(x, s_A^*(x))$ that lie on multiple equal utility curves. For most of values of x, $(x, s_A^*(x))$ lies on a unique equal utility curve, and the winning probability is $F_2[g(x)]$ no matter how the follower chooses between her best responses. So for the same s_A^*, the total expected utility of the leader under both Assumptions is the same as under only Assumption 2.2.

In conclusion, the optimal strategy of the leader under both Assumptions is the same as under only Assumption 2.2.

C.2. Assumption 2.2 is not necessary

When considering some tie-breaking rules other than always assigning the good to B, our method still works and the optimal strategy is the same.

We restate the situation: without Assumption 2.1, the follower may choose any best strategy now. We would like to prove that for arbitrary tie-breaking rules, the optimal strategy is the same as the optimal strategy under Assumption 2.2. The idea of the proof is that using the same strategy, leader A gets the same expected utility under arbitrary tie-breaking rules as under the tie-breaking rule under Assumption 2.2. Then we can prove leader A gets the same optimal expected utility under different tie-breaking rules.

First we should notice that without Assumption 2.2, the follower may not have a best response. To ensure the follower chooses the best response when tie-breaking rule assigns the good to A, we create t^+, i.e., $t + \epsilon$, where ϵ is extremely positive small. The follower can bid t^+ to represent a bid that arbitrarily approximate t, but with a weakly higher winning probability than t. Consider the following two settings.

(1) Under some other tie-breaking rule, A adopts strategy s_A.
(2) When the tie-breaking rule is assigning the good to B, A adopts strategy s_A.

Then in (1), B always has the best response. Suppose otherwise B does not have the best response with value y, then there is a series of bid $\{t_n\} \to t$, that approaches the largest utility $u_B(y)$.

$$\begin{aligned} u_B(y) &= \lim(y - p^w(t_n))P_B[t_n] - p^p(t_n) \\ &= (y - p^w(\lim t_n)) \lim P_B[t_n] - p^p(\lim t_n) \\ &= (y - p^w(t))P_B[t] - p^p(t) \text{ or } (y - p^w(t^+))P_B[t^+] - p^p(t^+) \end{aligned}$$

So either t or t^+ will become the best response.

Without loss of generality, we can assume s_A is sorted in weakly increasing order. Compared to (1), B has an advantage in (2), and $u_B^{(2)}(y) \geq u_B^{(1)}(y)$. (By $u_B^{(1)}$, we mean the follower's utility function in setting (1)). The follower could bid t^+ in (2) to guarantee the same utility as bidding t in (1). So we have $u_B^{(2)}(y) = u_B^{(1)}(y)$ and

(23) $\quad t \in S_B^{(2)}(y) \Rightarrow t^+ \in S_B^{(1)}(y)$

Here, $S_B^{(2)}(y)$ and $S_B^{(1)}(y)$ denote the follower's best response set in scenarios (2) and (1), respectively.

Since we introduce t^+ here, we need a new version of Lemma 4.4.

Theorem C.1 *For B's valuations $y_1 < y_2$, if $\exists a \in S_B^{(1)}(y_1), b \in S_B^{(1)}(y_2)$ but $a > b$, then either $a = b^+$, $P_B^{(1)}[b^+] = P_B^{(1)}[b]$ or $S_B^{(1)}(y_1) \subseteq S_B^{(1)}(y_2)$, $u_B^{(1)}(y_1) = u_B^{(1)}(y_2) = 0$, $P_B^{(1)}[b] = P_B^{(1)}[a] = 0$.*

We omit the proof of this theorem, because the proof is exactly the same.

By Theorem C.1, we can prove that conditional on $u_B(y) > 0$, the number of B's types that has multiple best responses is countable under any tie-breaking rule. So we can omit the discussion about how the follower responses in these types. We focus on the follower's types that has unique best response under both tie-breaking rules and also the follower's types that has zero expected utility. We find that even when the follower's responses change, they don't affect the leader A's winning probability.

- When $u_B(y) > 0$, if B's best response in (2), t, is an atom bid of the leader A, then she must bid t^+ in (1).

- When $u_B(y) > 0$, if B's best response in (2), t, is not an atom bid of the leader A, then she must bid t in (1).

When $u_B(y) = 0$, and there is no constant interval in s_A^* at the lowest bid $s_A(a_1)$, B will not bid more than $s_A(a_1)$ in both (1) and (2). The probability of tie is zero.

When $u_B(y) = 0$, and there is a constant interval in s_A^* at the lowest bid $s_A(a_1)$, there are several cases to consider,

1. if $y < p^w(s_A(a_1))$ and the follower bids lower than $s_A(a_1)$ in (2), then the follower might bid lower than $s_A(a_1)$ in (1), the leader's winning probability does not change.
2. if $y < p^w(s_A(a_1))$ and the follower bids lower than $s_A(a_1)$ in (2), then the follower might bid $s_A(a_1)$ in (1), we can prove the winner must be A when there is a tie.
3. when $y = p^w(s_A(a_1))$, the leader's winning probability does not change.
4. when $y > p^w(s_A(a_1))$, let B's best response in (2) be t, we have

$$(y - p^w(t))P_B[t] - p^p(t) = 0$$

   (a) Case: $t > s_A(a_1)$, we can prove that $p^p[t] = 0$ leads to contradiction. So $p^p[t] > 0$, we have $y > p^w(t)$. Then for any $y_0 > y$, we have $u_B(y_0) > 0$, so $y = \max\{y_0 | u_B(y_0) = 0\}$ is unique.

   (b) Case: $t = s_A(a_1)$, $(y - p^w(s_A(a_1)))P_B[t] - p^p[t] = 0$, y is unique.

   (c) Case: $t < s_A(a_1)$, $p^p[t] = 0$, we consider the best response of the follower, $\hat{t}$, in (1). There are three cases between $\hat{t}$ and $s_A(a_1)$. We can prove the the leader A's winning probability has no difference in each case and the proof is similar.

Now we have proved that the leader A gets the same optimal expected utility under different tie-breaking rules. Then we prove under different tie-breaking rules, the optimal expected utility is the same.

1. Under some specific tie-breaking rule, A uses the optimal strategy $\hat{s_A}$.

2. When some tie-breaking rule is assigning the good to B, A uses the optimal strategy s_A.
3. Under some specific tie-breaking rule, A uses the strategy s_A.
4. When some tie-breaking rule is assigning the good to B, A uses the strategy $\hat{s_A}$.

By our proof, we have (1) = (4) and (2) = (3). It's obvious that (1) $\geq$ (3) and (2) $\geq$ (4). So the optimal expected utility is the same under different tie-breaking rules.