Delay Reduction via Lagrange Multipliers in Stochastic Network Optimization

Longbo Huang, Michael J. Neely

***Abstract*—In this paper, we consider the problem of reducing network delay in stochastic network utility optimization problems. We start by studying the recently proposed quadratic Lyapunov function based algorithms (QLA, also known as the MaxWeight algorithm). We show that for every stochastic problem, there is a corresponding *deterministic* problem, whose dual optimal solution "exponentially attracts" the network backlog process under QLA. In particular, the probability that the backlog vector under QLA deviates from the attractor is exponentially decreasing in their Euclidean distance. This is the first such result for the class of algorithms built upon quadratic Lyapunov functions. The result quantifies the "network gravity" role of Lagrange Multipliers in network scheduling. It not only helps to explain how QLA achieves the desired performance but also suggests that one can roughly "subtract out" a Lagrange multiplier from the system induced by QLA.**

Based on this finding, we develop a family of *Fast Quadratic Lyapunov based Algorithms* (FQLA), which use virtual place-holder bits and virtual control processes for decision making. We prove that FQLA achieves an $[O(1/V), O([\log(V)]^2)]$ performance-delay tradeoff for problems with discrete action sets, and achieves a square-root tradeoff for continuous problems. The performance of FQLA is similar to the optimal tradeoffs achieved in prior work by Neely (2007) via drift-steering methods, and shows that QLA can also be used to approach such performance.

***Index Terms*—Queueing, Dynamic Control, Lyapunov analysis, Stochastic Optimization**

I. INTRODUCTION

In this paper, we consider the problem of reducing network delay in the following general framework of the stochastic network utility optimization problem. We are given a time slotted stochastic network. The network state, such as the network channel condition, is time varying according to some probability law. A network controller performs some action based on the observed network state at every time slot. The chosen action incurs a cost (since cost minimization is mathematically equivalent to utility maximization, below we will use cost and utility interchangeably), but also serves some amount of traffic and possibly generates new traffic for the network. This traffic causes congestion, and thus leads to backlogs at nodes in the network. The goal of the controller is to minimize its time average cost subject to the constraint that the time average total backlog in the network is finite.

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This work was presented in part at the 7th Intl. Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt), Seoul, June, 2009.

This material is supported in part by one or more of the following: the DARPA IT-MANET program grant W911NF-07-0028, the NSF grant OCE 0520324, the NSF Career grant CCF-0747525.

This setting is very general, and many existing works fall into this category. Further, many techniques have been used to study this problem (see [1] for a survey). In this paper, we focus on algorithms that are built upon quadratic Lyapunov functions (called QLA in the following), e.g., [2], [3], [4], [5], [6], [7]. These QLA algorithms are easy to implement, greedy in nature, and are parameterized by a scalar control variable V. It has been shown that when the network state is i.i.d., QLA algorithms can achieve a time average utility that is within $O(1/V)$ to the optimal. Therefore, as V grows large, the time average utility can be pushed arbitrarily close to the optimal. However, such close-to-optimal utility is usually at the expense of large network delay. In fact, in [3], [4], [7], it is shown that an $O(V)$ network delay is incurred when an $O(1/V)$ close-to-optimal utility is achieved. Two recent papers [8] and [9], which show that it is possible to achieve within $O(1/V)$ of optimal utility with only $O(\log(V))$ delay, use a more sophisticated algorithm design approach based on exponential Lyapunov functions. Therefore, it seems that though being simple in implementation, QLA algorithms have undesired delay performance.

However, we note that the delay results of QLA are usually given in terms of long term upper bounds of the average network backlog e.g., [7]. Thus they do not examine the possibility that the actual backlog vector (or its time average) converges to some fixed value. Work in [10] considers drift properties towards an "invariant" backlog vector, derived in the special case of a one-hop downlink system and when the problem exhibits a unique optimal Lagrange multiplier. An upper bound on the long term deviation of the actual backlog and the Lagrange multiplier vector is obtained. While this suggests Lagrange multipliers are "gravitational attractors," the bounds there do not show that the the actual backlog is very unlikely to deviate significantly from the attractor.

In this paper, we focus on obtaining stronger probability results of the steady state backlog process behavior under QLA. We first show that under QLA, even though the backlog can grow linearly in V, it "typically" stays close to an "attractor," which is the dual optimal solution of a *deterministic* optimization problem. In particular, the probability that the backlog vector deviates from the attractor is exponentially decreasing in distance. It significantly tightens the attractor analysis in [10]. It also implies that a large amount of the data is kept in the network simply for maintaining the backlog at the "right" level. Therefore, we can replace these data with some fake data (denoted as *place-holder bits* [11]) without heavily affecting QLA's performance. Based on this finding, we propose the *Fast Quadratic Lyapunov based Algorithms* (FQLA), which can be intuitively viewed as subtracting out a Lagrange multiplier from the system induced by QLA. We show that when the network state is i.i.d., FQLA is able to achieve within $O(1/V)$ of optimal utility with an $O([\log(V)]^2)$ delay guarantee for problems with discrete action sets, and achieve an $[O(1/V), O([\log(V)]^2\sqrt{V})]$ tradeoff for problems with a set of continuous action options. The development of FQLA also provides additional insights into QLA and the role of Lagrange multipliers in stochastic network optimization.

FQLA is closely related to the TOCA algorithm in [8], which obtains the same logarithmic and square-root tradeoffs for the energy-delay problem (up to a $\log(V)$ difference) via drift steering techniques. However, we note that FQLA differs from TOCA in the following: First, TOCA is constructed based on exponential Lyapunov functions; while FQLA uses simpler quadratic Lyapunov functions. Second, FQLA is designed to mimic QLA and is thus related to dual subgradient algorithms; whereas TOCA is designed to ensure the primal constraints are satisfied. Third, FQLA requires an arbitrary small but nonzero fraction of packet droppings, hence can not be applied to problems where packet dropping is not allowed.

We now summarize the main contributions of this paper:

(1) This paper proves that in steady state, the backlog vector process under QLA is "exponentially attracted" to an attractor, which is the dual optimal solution of a deterministic optimization problem. This is the first such result for the class of widely used QLA (MaxWeight) algorithms. This exponential attraction result quantifies the "network gravity" role of Lagrange multipliers in network scheduling and helps to explain how QLA achieves the desired performance. It is also the first theoretical result in the literature that explains the recent delay improvement result observed in [12], where by using LIFO together with QLA, one achieves a significant (around 90%) network delay reduction.

(2) This paper proposes the *Fast Quadratic Lyapunov based Algorithms (FQLA)* to "subtract out" a Lagrange multiplier from the network under QLA for delay reduction. FQLA is usually easy to implement, and can achieve an $\big[O(1/V), O([\log(V)]^2)\big]$ performance-delay tradeoff for general stochastic network optimization problems with a discrete set of action options as well as a square-root tradeoff for continuous problems.

The paper is organized as follows: In Section II, we set up our notations. In Section III, we state our network model. We then review the QLA algorithm and define the *deterministic problem* in Section IV. In Section V, we show that the backlog process under QLA always stays close to an attractor. In Section VI, we propose the FQLA algorithm. Section VII provides simulation results.

II. Notations

- $\mathbb{R}$ ($\mathbb{R}_+$ or $\mathbb{R}_-$): the set of (nonnegative or non-positive) real numbers
- $\mathbb{R}^n$ (or $\mathbb{R}^n_+$): the set of n dimensional *column* vectors, with each element being in $\mathbb{R}$ (or $\mathbb{R}_+$)
- **bold** symbols $\boldsymbol{a}$ and $\boldsymbol{a}^T$: *column* vector and its transpose
- $\boldsymbol{a} \succeq \boldsymbol{b}$: vector $\boldsymbol{a}$ is entrywise no less than vector $\boldsymbol{b}$
- $||\boldsymbol{a} - \boldsymbol{b}||$: the Euclidean distance of $\boldsymbol{a}$ and $\boldsymbol{b}$

III. System Model

In this section, we specify the general network model we use. We consider a network controller that operates a network with the goal of minimizing the time average cost, subject to the queue stability constraint. The network is assumed to operate in slotted time, i.e., $t \in \{0, 1, 2, ...\}$. We assume there are $r \geq 1$ queues in the network.

A. Network State

We assume there are a total of M different random network states, and define $\mathcal{S} = \{s_1, s_2, \dots, s_M\}$ as the set of possible states. Each particular state s_i indicates the current network parameters, such as a vector of channel conditions for each link, or a collection of other relevant information about the current network channels and arrivals. Let $S(t)$ denote the network state at time t. We assume that $S(t)$ is i.i.d. every time slot, and let p_{s_i} denote its probability of being in state s_i, i.e., $p_{s_i} = Pr\{S(t) = s_i\}$. We assume the network controller can observe $S(t)$ at the beginning of every slot t, but the p_{s_i} probabilities are not necessarily known. Note that if $S(t)$ contains multiple components, e.g., if $S(t)$ is a vector of channel states of the network, the components can be correlated to each other. See Section III-D for an example.

B. The Cost, Traffic, and Service

At each time t, after observing $S(t) = s_i$, the controller chooses an action $x(t)$ from a set $\mathcal{X}^{(s_i)}$, i.e., $x(t) = x^{(s_i)}$ for some $x^{(s_i)} \in \mathcal{X}^{(s_i)}$. The set $\mathcal{X}^{(s_i)}$ is called the feasible action set for network state s_i and is assumed to be time-invariant and compact for all $s_i \in \mathcal{S}$. The cost, traffic, and service generated by the chosen action $x(t) = x^{(s_i)}$ are as follows:

(a) The chosen action has an associated cost given by the cost function $f(t) = f(s_i, x^{(s_i)}) : \mathcal{X}^{(s_i)} \mapsto \mathbb{R}_+$ (or $\mathcal{X}^{(s_i)} \mapsto \mathbb{R}_-$ in reward maximization problems);

(b) The amount of traffic generated by the action to queue j is determined by the traffic function $A_j(t) = A_j(s_i, x^{(s_i)}) : \mathcal{X}^{(s_i)} \mapsto \mathbb{R}_+$, in units of packets;

(c) The amount of service allocated to queue j is given by the rate function $\mu_j(t) = \mu_j(s_i, x^{(s_i)}) : \mathcal{X}^{(s_i)} \mapsto \mathbb{R}_+$, in units of packets;

Note that $A_j(t)$ includes both the exogenous arrivals from outside the network to queue j, and the endogenous arrivals from other queues, i.e., the transmitted packets from other queues, to queue j (See Section III-C and III-D for further explanations). We assume the functions $f(s_i, \cdot)$, $\mu_j(s_i, \cdot)$ and $A_j(s_i, \cdot)$ are time-invariant, their magnitudes are uniformly upper bounded by some constant $\delta_{max} \in (0, \infty)$ for all s_i, j, and they are known to the network operator. We also assume that there exists a set of actions $\{x^{(s_i)k}\}_{i=1,\dots,M}^{k=1,2,\dots,\infty}$ with $x^{(s_i)k} \in \mathcal{X}^{(s_i)}$ such that $\sum_{s_i} p_{s_i}\big\{\sum_k \vartheta_k^{(s_i)}[A_j(s_i, x^{(s_i)k}) - \mu_j(s_i, x^{(s_i)k})]\big\} \leq -\epsilon$ for some $\epsilon > 0$ for all j, with $\sum_k \vartheta_k^{(s_i)} = 1$ and $\vartheta_k^{(s_i)} \geq 0$ for all s_i and k. That is, the constraints are feasible with ϵ slackness. Thus, there exists a stationary randomized policy that stabilizes all queues (where

$\vartheta_k^{(s_i)}$ represents the probability of choosing action $x^{(s_i)k}$ when $S(t) = s_i$). In the following, we use:

$$\boldsymbol{A}(t) = (A_1(t), ..., A_r(t))^T, \ \ \boldsymbol{\mu}(t) = (\mu_1(t), ..., \mu_r(t))^T, \quad (1)$$

to denote the arrival and service vectors at time t. It is easy to see from above that if we define:

$$B = \sqrt{r}\delta_{max}, \quad (2)$$

then $\|\boldsymbol{A}(t) - \boldsymbol{\mu}(t)\| \leq B$ for all t.

C. Queueing, Average Cost, and the Stochastic Problem

Let $\boldsymbol{q}(t) = (q_1(t), ..., q_r(t))^T \in \mathbb{R}_+^r, \ t = 0, 1, 2, ...$ be the queue backlog vector process of the network, in units of packets. We assume the following queueing dynamics:

$$q_j(t+1) = \max\big[q_j(t) - \mu_j(t), 0\big] + A_j(t) \quad \forall j, \quad (3)$$

and $\boldsymbol{q}(0) = \boldsymbol{0}$. By using (3), we assume that when a queue does not have enough packets to send, null packets are transmitted. In this paper, we adopt the following notion of queue stability:

$$\mathbb{E}\big\{\sum_{j=1}^{r} q_j\big\} \triangleq \limsup_{t\rightarrow\infty} \frac{1}{t}\sum_{\tau=0}^{t-1}\sum_{j=1}^{r}\mathbb{E}\big\{q_j(\tau)\big\} < \infty. \quad (4)$$

Note that this criterion is not restrictive. It can be shown that our results remain the same under a wide class of other stability criterions used in the literature. For more details, see [13] and [14]. We also use f_{av}^{π} to denote the time average cost induced by an action-seeking policy π, defined as:

$$f_{av}^{\pi} \triangleq \limsup_{t\rightarrow\infty} \frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}\big\{f^{\pi}(\tau)\big\}, \quad (5)$$

where $f_{av}^{\pi}(\tau)$ is the cost incurred at time τ by policy π. We call an action-seeking policy *feasible* if at every time slot t it only chooses actions from the feasible action set $\mathcal{X}^{(S(t))}$. We then call a feasible action-seeking policy under which (4) holds a *stable* policy, and use f_{av}^* to denote the optimal time average cost over all stable policies. In every slot, the network controller observes the current network state and chooses a control action, with the goal of minimizing time average cost subject to network stability. This goal can be mathematically stated as: **(P1)** $\ \boldsymbol{\min: \ f_{av}^{\pi}, \ \ s.t. (4)}$. In the following, we will refer to **(P1)** as *the stochastic problem*. This stochastic problem framework can be used to model many network problems, e.g., the energy minimization problem [3] and the access point pricing problem [5]. We note that a similar network model with stochastic penalties is treated in [15] using a fluid model and a primal-dual approach that achieves optimality in a limiting sense. The framework is also treated in [7] using a quadratic Lyapunov based algorithm (QLA) that provides an explicit $[O(1/V), O(V)]$ performance-delay tradeoff when the network state is i.i.d..

D. An Example of the Model

Here we provide an example to illustrate our model. Consider the 2-queue network in Fig.1. In every slot, the network operator decides whether or not to allocate one unit power to serve packets at each queue, so as to support all arriving traffic, i.e., maintain queue stability, with minimum energy expenditure. The number of arrival packets $R(t)$, is i.i.d. over slots, being either 2 or 0 with probabilities $5/8$ and $3/8$ respectively. Each channel state $CH_1(t)$ or $CH_2(t)$ can be either "G=good" or "B=bad." However, the two channels are correlated, so that $(CH_1(t), CH_2(t))$ can only be in the channel set $\mathcal{CH} = \{(B,B), (B,G), (G,G)\}$. We assume $(CH_1(t), CH_2(t))$ is i.i.d. over slots and takes any value in $\mathcal{CH}$ with probability $\frac{1}{3}$. When a link's channel state is good, one unit of power can serve 2 packets over the link, otherwise it can only serve one. We assume powers can be allocated to both channels without affecting each other.

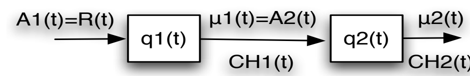

Fig. 1. A 2-queue system

In this case, the network state $S(t)$ is a triple $(R(t), CH_1(t), CH_2(t))$ and is i.i.d.. There are six possible network states. At each state s_i, the action $x^{(s_i)}$ is a pair (x_1, x_2), with x_i being the amount of energy spent at queue i, and $(x_1, x_2) \in \mathcal{X}^{(s_i)} = \{0/1, 0/1\}$. The cost function is $f(s_i, x^{(s_i)}) = x_1 + x_2, \forall s_i$. The network states, the traffic functions, and the service rate functions are summarized in Fig. 2. Note here $A_1(t) = R(t)$ is part of $S(t)$ and is independent of $x^{(s_i)}$; while $A_2(t) = \mu_1(t)$ hence depends on $x^{(s_i)}$. Also note that $A_2(t)$ equals $\mu_1(t)$ instead of $\min[\mu_1(t), q_1(t)]$ due to our idle fill assumption in Section III-C.

S(t)	$R(t)$	$CH_1(t)$	$CH_2(t)$	$A_1(t)$	$A_2(t)$	$\mu_1(t)$	$\mu_2(t)$
s_1	0	B	B	0	x_1	x_1	x_2
s_2	0	B	G	0	x_1	x_1	$2x_2$
s_3	0	G	G	0	$2x_1$	$2x_1$	$2x_2$
s_4	2	B	B	2	x_1	x_1	x_2
s_5	2	B	G	2	x_1	x_1	$2x_2$
s_6	2	G	G	2	$2x_1$	$2x_1$	$2x_2$

Fig. 2. Network state, Traffic, and Rate functions

IV. QLA and the Deterministic Problem

In this section, we first review the quadratic Lyapunov functions based algorithms (the QLA algorithm) [7] for solving the stochastic problem. Then we define the *deterministic problem* and its dual problem. We then also discuss some properties of the dual function. The dual problem and the properties of the dual function will be used later for our analysis of the steady state backlog behavior under QLA.

A. The QLA algorithm

To solve the stochastic problem using QLA, we first define a quadratic Lyapunov function $L(\boldsymbol{q}(t)) = \frac{1}{2}\sum_{j=1}^{r} q_j^2(t)$. We then define the one-slot conditional Lyapunov drift: $\Delta(\boldsymbol{q}(t)) = \mathbb{E}\big\{L(\boldsymbol{q}(t+1)) - L(\boldsymbol{q}(t)) \mid \boldsymbol{q}(t)\big\}$, where the expectation is taken over the random network state $S(t)$ and the possible random actions. From (3), we obtain the following:

$$\Delta(\boldsymbol{q}(t)) \leq B^2 - \mathbb{E}\big\{\sum_{j=1}^{r} q_j(t)\big[\mu_j(t) - A_j(t)\big] \mid \boldsymbol{q}(t)\big\}.$$

Now add to both sides the term $V\mathbb{E}\{f(t) \mid \boldsymbol{q}(t)\}$, where $V \geq 1$ is a scalar control variable, we obtain:

$$\Delta(\boldsymbol{q}(t)) + V\mathbb{E}\{f(t) \mid \boldsymbol{q}(t)\} \leq B^2 - \mathbb{E}\Big\{ - Vf(t) + \sum_{j=1}^{r} q_j(t)\big[\mu_j(t) - A_j(t)\big] \mid \boldsymbol{q}(t)\Big\}. \quad (6)$$

The QLA algorithm is then obtained by choosing an action x at every time t to *minimize the right hand side of (6)* given $\boldsymbol{q}(t)$. Specifically, the QLA algorithm works as follows: [1]

QLA: At every time slot t, observe the current network state $S(t)$ and the backlog $\boldsymbol{q}(t)$. If $S(t) = s_i$, choose $x^{(s_i)} \in \mathcal{X}^{(s_i)}$ that solves the following:

$$\begin{aligned} \max: \quad & -Vf(s_i, x) + \sum_{j=1}^{r} q_j(t)\big[\mu_j(s_i, x) - A_j(s_i, x)\big] \quad (7) \\ s.t. \quad & x \in \mathcal{X}^{(s_i)}. \end{aligned}$$

Depending on the problem structure, (7) can usually be decomposed into separate parts that are easier to solve, e.g., [3], [5]. Also, it can be shown, as in [7] that,

$$f_{av}^{QLA} = f_{av}^{*} + O(1/V), \quad \overline{q}^{QLA} = O(V), \quad (8)$$

where f_{av}^{QLA} and $\overline{q}^{QLA}$ are the expected average cost and the expected average network backlog under QLA, respectively.

B. The Deterministic Problem

Consider *the deterministic problem* as follows:

$$\begin{aligned} \min: \quad & \mathcal{F}(\boldsymbol{x}) \triangleq V\sum_{s_i} p_{s_i} f(s_i, x^{(s_i)}) \quad (9) \\ s.t. \quad & \mathcal{A}_j(\boldsymbol{x}) \triangleq \sum_{s_i} p_{s_i} A_j(s_i, x^{(s_i)}) \\ & \qquad \leq \mathcal{B}_j(\boldsymbol{x}) \triangleq \sum_{s_i} p_{s_i}\mu_j(s_i, x^{(s_i)}) \quad \forall j \\ & x^{(s_i)} \in \mathcal{X}^{(s_i)} \quad \forall i = 1, 2, ..., M, \end{aligned}$$

where p_{s_i} corresponds to the probability of $S(t) = s_i$ and $\boldsymbol{x} = (x^{(s_1)}, ..., x^{(s_M)})^T$. The dual problem of (9) can be obtained as follows ($\mathbf{0} \in \mathbb{R}^r$ has all entries being 0):

$$\max: \ g(\boldsymbol{\gamma}), \quad s.t., \ \boldsymbol{\gamma} \succeq \mathbf{0}, \quad (10)$$

where $g(\boldsymbol{\gamma})$ is called the dual function and is defined as:

$$g(\boldsymbol{\gamma}) = \inf_{x^{(s_i)} \in \mathcal{X}^{(s_i)}} \sum_{s_i} p_{s_i}\Big\{Vf(s_i, x^{(s_i)}) + \sum_j \gamma_j\big[A_j(s_i, x^{(s_i)}) - \mu_j(s_i, x^{(s_i)})\big]\Big\}. \quad (11)$$

Here $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_r)^T$ is the *Lagrange multiplier* of (9). It is well known that $g(\boldsymbol{\gamma})$ in (11) is concave in the vector $\boldsymbol{\gamma}$, and hence the problem (10) can usually be solved efficiently, particularly when cost functions and rate functions are separable over different network components. It is also well known that in many situations, the optimal value of (10) is the same as the optimal value of (9) and in this case we say that there is no duality gap [16].

We note that the deterministic problem (9) is not necessarily convex as the sets $\mathcal{X}^{(s_i)}$ are not necessarily convex, and the functions $f(s_i, \cdot)$, $A_j(s_i, \cdot)$ and $\mu_j(s_i, \cdot)$ are not necessarily convex. Therefore, there may be a duality gap between the deterministic problem (9) and its dual (10). Furthermore, solving the deterministic problem (9) may not solve the stochastic problem. This is so since at every network state, the stochastic problem may require time sharing over more than one action, but the solution to the deterministic problem gives only a fixed operating point per network state. However, one can show that the dual problem (10) gives the exact value of Vf_{av}^{*}, where f_{av}^{*} is the optimal time average cost for the stochastic problem, even if (9) is non-convex.

For a given $\boldsymbol{\gamma}$, let $\boldsymbol{x}_{\boldsymbol{\gamma}} = (x_{\boldsymbol{\gamma}}^{(s_1)}, x_{\boldsymbol{\gamma}}^{(s_2)}, ..., x_{\boldsymbol{\gamma}}^{(s_M)})^T$ with $x_{\boldsymbol{\gamma}}^{(s_i)} \in \mathcal{X}^{(s_i)}, \forall i$, be a minimizer of the right-hand side of $g(\boldsymbol{\gamma})$. The term $\boldsymbol{G}_{\boldsymbol{\gamma}} = (G_{\boldsymbol{\gamma},1}, G_{\boldsymbol{\gamma},2}, ..., G_{\boldsymbol{\gamma},r})^T$ with:

$$\begin{aligned} G_{\boldsymbol{\gamma},j} &= \mathcal{A}_j(\boldsymbol{x}_{\boldsymbol{\gamma}}) - \mathcal{B}_j(\boldsymbol{x}_{\boldsymbol{\gamma}}) \quad (12) \\ &= \sum_{s_i} p_{s_i}\big[-\mu_j(s_i, x_{\boldsymbol{\gamma}}^{(s_i)}) + A_j(s_i, x_{\boldsymbol{\gamma}}^{(s_i)})\big], \end{aligned}$$

is then called the *subgradient* of $g(\cdot)$ at $\boldsymbol{\gamma}$ [16]. It is well known that for any other $\hat{\boldsymbol{\gamma}} \in \mathbb{R}^r$, we have:

$$(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})^T\boldsymbol{G}_{\boldsymbol{\gamma}} \geq g(\hat{\boldsymbol{\gamma}}) - g(\boldsymbol{\gamma}). \quad (13)$$

Using $\|\boldsymbol{G}_{\boldsymbol{\gamma}}\| \leq B$, we note that (13) also implies:

$$g(\hat{\boldsymbol{\gamma}}) - g(\boldsymbol{\gamma}) \leq B\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\| \quad \forall\, \hat{\boldsymbol{\gamma}}, \boldsymbol{\gamma} \in \mathbb{R}^r \quad (14)$$

We are now ready to study the steady state behavior of $\boldsymbol{q}(t)$ under QLA. To simplify notations and highlight the scaling effect of the scalar V in QLA, we use $g_0(\boldsymbol{\gamma})$ and $\boldsymbol{\gamma}_0^*$ to denote the dual objective function and an optimal solution of (10) when $V = 1$; and use $g(\boldsymbol{\gamma})$ and $\boldsymbol{\gamma}_V^*$ (also called the optimal Lagrange multiplier) for their counterparts with general $V \geq 1$. To simplify analysis, we assume the following throughout:

***Assumption** 1:* $\boldsymbol{\gamma}_V^* = (\gamma_{V1}^*, ..., \gamma_{Vr}^*)^T$ is unique $\forall\ V \geq 1$.

Note that Assumption 1 is not very restrictive. In fact, it holds in many network utility optimization problems, e.g., [10]. In many cases, we also have $\boldsymbol{\gamma}_V^* \neq \mathbf{0}$. Moreover, for the assumption to hold for all $V \geq 1$, it suffices to have just $\boldsymbol{\gamma}_0^*$ being unique. This is shown in the following lemma.

***Lemma** 1:* $\boldsymbol{\gamma}_V^* = V\boldsymbol{\gamma}_0^*$.

Proof: From (11) we see that:

$$g(\boldsymbol{\gamma})/V = \inf_{x^{(s_i)} \in \mathcal{X}^{(s_i)}} \sum_{s_i} p_{s_i}\Big\{f(s_i, x^{(s_i)}) + \sum_j \hat{\gamma}_j\big[A_j(s_i, x^{(s_i)}) - \mu_j(s_i, x^{(s_i)})\big]\Big\},$$

where $\hat{\gamma}_j = \frac{\gamma_j}{V}$. The right hand side is exactly $g_0(\hat{\boldsymbol{\gamma}})$, and so is maximized at $\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_0^*$. Thus $g(\boldsymbol{\gamma})$ is maximized at $V\boldsymbol{\gamma}_0^*$. ■

[1] We assume without loss of generality that an optimal solution of (7) exists. This condition can easily be satisfied if all $f(s_i, \cdot)$, $\mu_j(s_i, \cdot)$ and $A_j(s_i, \cdot)$ functions are continuous.

V. Backlog vector behavior under QLA

In this section we study the backlog vector behavior under QLA of the stochastic problem. We first look at the case when $g_0(\boldsymbol{\gamma})$ is "locally polyhedral." We show that $\boldsymbol{q}(t)$ is mostly within $O(\log(V))$ distance from $\boldsymbol{\gamma}_V^*$ in this case, even when $S(t)$ evolves according to a more general time homogeneous Markovian process. We then consider the case when $g_0(\boldsymbol{\gamma})$ is "locally smooth," and show that $\boldsymbol{q}(t)$ is mostly within $O(\sqrt{V}\log(V))$ distance from $\boldsymbol{\gamma}_V^*$. As we will see, these two results also explain how QLA functions. The choices of these two types of $g_0(\boldsymbol{\gamma})$ functions are made based on their practical generality. See Section V-C for further discussion.

A. When $g_0()$ is "locally polyhedral"

In this section, we study the backlog vector behavior under QLA for the case where $g_0(\boldsymbol{\gamma})$ is *locally polyhedral* with parameters ϵ, L, i.e., there exist $\epsilon, L > 0$, such that for all $\boldsymbol{\gamma} \succeq \mathbf{0}$ with $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0^*\| < \epsilon$, the dual function $g_0(\boldsymbol{\gamma})$ satisfies:

$$g_0(\boldsymbol{\gamma}_0^*) \geq g_0(\boldsymbol{\gamma}) + L\|\boldsymbol{\gamma}_0^* - \boldsymbol{\gamma}\|. \tag{15}$$

We will show that in this case, even if $S(t)$ is a general time homogeneous Markovian process, the backlog vector will mostly be within $O(\log(V))$ distance to $\boldsymbol{\gamma}_V^*$. Hence the same is also true when $S(t)$ is i.i.d..

To start, we assume *for this subsection* that $S(t)$ evolves according to a time homogeneous Markovian process. Now we define the following notations. Given t_0, define $\mathcal{T}_{s_i}(t_0, k)$ to be the set of slots at which $S(\tau) = s_i$ for $\tau \in [t_0, t_0+k-1]$. For a given $\nu > 0$, define the *convergent interval* T_ν [17] for the $S(t)$ process to be the smallest number of slots such that for any t_0, regardless of past history, we have:

$$\sum_{i=1}^{M} \left| p_{s_i} - \frac{\mathbb{E}\big\{||\mathcal{T}_{s_i}(t_0, T_\nu)|| \mid \mathcal{H}(t_0)\big\}}{T_\nu} \right| \leq \nu, \tag{16}$$

here $||\mathcal{T}_{s_i}(t_0, T_\nu)||$ is the cardinality of $\mathcal{T}_{s_i}(t_0, T_\nu)$, and $\mathcal{H}(t_0) = \{S(\tau)\}_{\tau=0}^{t_0-1}$ denotes the network state history up to time t_0. For any $\nu > 0$, such a T_ν must exist for any stationary ergodic processes with finite state space, thus T_ν exists for $S(t)$ in particular. When $S(t)$ is i.i.d. every slot, we have $T_\nu = 1$ for all $\nu \geq 0$, as $\mathbb{E}\big\{||\mathcal{T}_{s_i}(t_0, 1)|| \mid \mathcal{H}(t_0)\big\} = p_{s_i}$. Intuitively, T_ν represents the time needed for the process to reach its "near" steady state.

The following theorem summarizes the main results. Recall that B is defined in (2) as the upper bound of the magnitude change of $\boldsymbol{q}(t)$ in a slot, which is a function of the network size r and δ_{max}.

***Theorem** 1:* If $g_0(\boldsymbol{\gamma})$ is locally polyhedral with constants $\epsilon, L > 0$, independent of V, then under QLA,

(a) There exist constants $\nu > 0$, $D \geq \eta > 0$, all independent of V, such that $D = D(\nu), \eta = \eta(\nu)$, and whenever $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| \geq D$, we have:

$$\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\| \mid \boldsymbol{q}(t)\big\} \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| - \eta. \tag{17}$$

In particular, the constants ν, D and η that satisfy (17) can be chosen as follows: Choose ν as any constant such that $0 < \nu < L/B$. Then choose η as any value such that $0 < \eta < T_\nu(L - B\nu)$. Finally, choose D as: [2]

$$D = \max\left[\frac{(T_\nu^2 + T_\nu)B^2 - \eta^2}{2T_\nu(L - \frac{\eta}{T_\nu} - B\nu)}, \eta\right]. \tag{18}$$

(b) For given constants ν, D, η in (a), there exist some constants $c^*, \beta^* > 0$, independent of V, such that:

$$\mathcal{P}(D, m) \leq c^* e^{-\beta^* m}, \tag{19}$$

where $\mathcal{P}(D, m)$ is defined as:

$$\mathcal{P}(D, m) \triangleq \limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} Pr\{\|\boldsymbol{q}(\tau) - \boldsymbol{\gamma}_V^*\| > D + m\}. \tag{20}$$

Note that if $m = \frac{\log(V)}{\beta^*}$, by (19) we have $\mathcal{P}(D, m) \leq \frac{c^*}{V}$. Also if a steady state distribution of $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|$ exists under QLA, e.g., when $q_j(t)$ only takes integer values for all j, in which case $\boldsymbol{q}(t)$ is a discrete time Markov chain with countably infinite states and the limit of $\frac{1}{t}\sum_{\tau=0}^{t-1} Pr\{\|\boldsymbol{q}(\tau) - \boldsymbol{\gamma}_V^*\| > D + m\}$ exists as $t \to \infty$, then one can replace $\mathcal{P}(D, m)$ with the steady state probability that $\boldsymbol{q}(t)$ deviates from $\boldsymbol{\gamma}_V^*$ by an amount of $D + m$, i.e., $Pr\{\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| > D + m\}$. Therefore Theorem 1 can be viewed as showing that when (15) is satisfied, for a large V, the backlog $\boldsymbol{q}(t)$ under QLA will mostly be within $O(\log(V))$ distance from $\boldsymbol{\gamma}_V^*$. This implies that the average backlog will roughly be $\sum_j \gamma_{Vj}^*$, which is typically $\Theta(V)$ by Lemma 1. However, this fact will also allow us to build FQLA upon QLA to "subtract out" roughly $\sum_j \gamma_{Vj}^*$ data from the network and reduce network delay. Theorem 1 also highlights a deep connection between the steady state behavior of the network backlog $\boldsymbol{q}(t)$ and the structure of $g_0(\boldsymbol{\gamma})$. We also note that (15) is not very restrictive. In fact, if $g_0(\boldsymbol{\gamma})$ is polyhedral (e.g., $\mathcal{X}^{(s_i)}$ is finite for all s_i), with a unique optimal solution $\boldsymbol{\gamma}_0^* \succeq \mathbf{0}$, then (15) can usually be satisfied (see Section VII for an example). To prove the theorem, we need the following lemma.

***Lemma** 2:* For any $\nu > 0$, under QLA, we have for all t,

$$\begin{aligned}\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \mid \boldsymbol{q}(t)\big\}& \\ \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 + (T_\nu^2 + T_\nu)B^2& \\ -2T_\nu\big(g(\boldsymbol{\gamma}_V^*) - g(\boldsymbol{q}(t))\big) + 2T_\nu \nu B\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\|.&\end{aligned} \tag{21}$$

Proof: See Appendix A. ■

We now use Lemma 2 to prove Theorem 1.

Proof: (Theorem 1) Part (a): We first show that if (15) holds for $g_0(\boldsymbol{\gamma})$ with L, then it also holds for $g(\boldsymbol{\gamma})$ with the same L. To this end, suppose (15) holds for $g_0(\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma}$ satisfying $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0^*\| < \epsilon$. Then for any $\boldsymbol{\gamma} \succeq \mathbf{0}$ such that $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_V^*\| < \epsilon V$, we have $\|\boldsymbol{\gamma}/V - \boldsymbol{\gamma}_0^*\| < \epsilon$, hence:

$$g_0(\boldsymbol{\gamma}_0^*) \geq g_0(\boldsymbol{\gamma}/V) + L\|\boldsymbol{\gamma}_0^* - \boldsymbol{\gamma}/V\|.$$

Multiplying both sides by V, we get:

$$Vg_0(\boldsymbol{\gamma}_0^*) \geq Vg_0(\boldsymbol{\gamma}/V) + LV\|\boldsymbol{\gamma}_0^* - \boldsymbol{\gamma}/V\|.$$

Now using $\boldsymbol{\gamma}_V^* = V\boldsymbol{\gamma}_0^*$ and $g(\boldsymbol{\gamma}) = Vg_0(\boldsymbol{\gamma}/V)$, we have for all $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_V^*\| < \epsilon V$:

$$g(\boldsymbol{\gamma}_V^*) \geq g(\boldsymbol{\gamma}) + L\|\boldsymbol{\gamma}_V^* - \boldsymbol{\gamma}\|. \tag{22}$$

[2] It can be seen from (14) that $B \geq L$. Thus $T_\nu B > \eta$.

Since $g(\boldsymbol{\gamma})$ is concave, we see that (22) indeed holds for all $\boldsymbol{\gamma} \succeq \mathbf{0}$. Now for a given $\eta > 0$, if:

$$(T_\nu^2 + T_\nu)B^2 - 2T_\nu\big(g(\boldsymbol{\gamma}_V^*) - g(\boldsymbol{q}(t))\big) \quad (23)$$
$$+2T_\nu \nu B\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\| \leq \eta^2 - 2\eta\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\|,$$

then by (21), we have:

$$\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \mid \boldsymbol{q}(t)\big\} \leq (\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| - \eta)^2,$$

which then by Jensen's inequality implies:

$$(\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\| \mid \boldsymbol{q}(t)\big\})^2 \leq (\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| - \eta)^2.$$

Thus (17) follows whenever (23) holds and $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| \geq \eta$. It suffices to choose D and η such that $D \geq \eta$ and that (23) holds whenever $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| \geq D$. Now note that (23) can be rewritten as the following inequalty:

$$g(\boldsymbol{\gamma}_V^*) \geq g(\boldsymbol{q}(t)) + (B\nu + \frac{\eta}{T_\nu})\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\| + \mathcal{Y}, \quad (24)$$

where $\mathcal{Y} = \frac{(T_\nu^2 + T_\nu)B^2 - \eta^2}{2T_\nu}$. Choose any $\nu > 0$ independent of V such that $B\nu < L$ and choose $\eta \in (0, T_\nu(L - B\nu))$. By (22), if:

$$L\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| \geq (B\nu + \frac{\eta}{T_\nu})\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\| + \mathcal{Y} \quad (25)$$

then (24) holds. Now choose D as defined in (18), we see that if $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\| \geq D$, then (25) holds, which implies (24), and equivalently (23). We also have $D \geq \eta$, hence (17) holds.

Part (b): Now we show that (17) implies (19). Choose constants ν, D and η that are independent of V in (a). Denote $Y(t) = \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|$, we see then whenever $Y(t) \geq D$, we have $\mathbb{E}\big\{Y(t+T_\nu) - Y(t) \mid \boldsymbol{q}(t)\big\} \leq -\eta$. It is also easy to see that $|Y(t+T_\nu) - Y(t)| \leq T_\nu B$, as B is defined in (2) as the upper bound of the magnitude change of $\boldsymbol{q}(t)$ in a slot. Define $\tilde{Y}(t) = \max\big[Y(t) - D, 0\big]$. We see that whenever $\tilde{Y}(t) \geq T_\nu B$, we have:

$$\mathbb{E}\big\{\tilde{Y}(t+T_\nu) - \tilde{Y}(t) \mid \boldsymbol{q}(t)\big\} \quad (26)$$
$$= \mathbb{E}\big\{Y(t+T_\nu) - Y(t) \mid \boldsymbol{q}(t)\big\} \leq -\eta.$$

Now define a Lyapunov function of $\tilde{Y}(t)$ to be $L(\tilde{Y}(t)) = e^{w\tilde{Y}(t)}$ with some $w > 0$, and define the T_ν-slot conditional *drift* to be:

$$\Delta_{T_\nu}(\tilde{Y}(t)) \triangleq \mathbb{E}\big\{L(\tilde{Y}(t+T_\nu)) - L(\tilde{Y}(t)) \mid \boldsymbol{q}(t)\big\}$$
$$= \mathbb{E}\big\{e^{w\tilde{Y}(t+T_\nu)} - e^{w\tilde{Y}(t)} \mid \boldsymbol{q}(t)\big\}. \quad (27)$$

It is shown in Appendix B that by choosing $w = \frac{\eta}{T_\nu^2 B^2 + T_\nu B\eta/3}$, we have for all $\tilde{Y}(t) \geq 0$:

$$\Delta_{T_\nu}(\tilde{Y}(t)) \leq e^{2wT_\nu B} - \frac{w\eta}{2}e^{w\tilde{Y}(t)}. \quad (28)$$

Taking expectation on both sides, we have:

$$\mathbb{E}\big\{e^{w\tilde{Y}(t+T_\nu)} - e^{w\tilde{Y}(t)}\big\} \leq e^{2wT_\nu B} - \frac{w\eta}{2}\mathbb{E}\big\{e^{w\tilde{Y}(t)}\big\}. \quad (29)$$

Now summing (29) over $t \in \{t_0, t_0 + T_\nu, ..., t_0 + (N-1)T_\nu\}$ for some $t_0 \in \{0, 1, ..., T_\nu - 1\}$, we have:

$$\mathbb{E}\big\{e^{w\tilde{Y}(t_0+NT_\nu)} - e^{w\tilde{Y}(t_0)}\big\} \leq Ne^{2wT_\nu B}$$
$$- \sum_{j=0}^{N-1}\frac{w\eta}{2}\mathbb{E}\big\{e^{w\tilde{Y}(t_0+jT_\nu)}\big\}.$$

Rearranging the terms, we have:

$$\sum_{j=0}^{N-1}\frac{w\eta}{2}\mathbb{E}\big\{e^{w\tilde{Y}(t_0+jT_\nu)}\big\} \leq Ne^{2wT_\nu B} + \mathbb{E}\big\{e^{w\tilde{Y}(t_0)}\big\}.$$

Summing the above over $t_0 \in \{0, 1, ..., T_\nu - 1\}$, we obtain:

$$\sum_{t=0}^{NT_\nu - 1}\frac{w\eta}{2}\mathbb{E}\big\{e^{w\tilde{Y}(t)}\big\} \leq NT_\nu e^{2wT_\nu B} + \sum_{t_0=0}^{T_\nu - 1}\mathbb{E}\big\{e^{w\tilde{Y}(t_0)}\big\}.$$

Dividing both sides with NT_ν, we obtain:

$$\frac{1}{NT_\nu}\sum_{t=0}^{NT_\nu - 1}\frac{w\eta}{2}\mathbb{E}\big\{e^{w\tilde{Y}(t)}\big\} \leq e^{2wT_\nu B} \quad (30)$$
$$+\frac{1}{NT_\nu}\sum_{t_0=0}^{T_\nu - 1}\mathbb{E}\big\{e^{w\tilde{Y}(t_0)}\big\}.$$

Taking the limsup as N goes to infinity, we obtain:

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}\frac{w\eta}{2}\mathbb{E}\big\{e^{w\tilde{Y}(\tau)}\big\} \leq e^{2wT_\nu B}. \quad (31)$$

Using the fact that $\mathbb{E}\big\{e^{w\tilde{Y}(\tau)}\big\} \geq e^{wm}Pr\{\tilde{Y}(\tau) > m\}$,

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}\frac{w\eta}{2}e^{wm}Pr\{\tilde{Y}(\tau) > m\} \leq e^{2wT_\nu B}. \quad (32)$$

Plug in $w = \frac{\eta}{T_\nu^2 B^2 + T_\nu B\eta/3}$ and use the definition of $\tilde{Y}(t)$:

$$\mathcal{P}(D, m) \leq \frac{2e^{2wT_\nu B}}{w\eta}e^{-wm} \quad (33)$$
$$= \frac{2(T_\nu^2 B^2 + T_\nu B\eta/3)e^{\frac{2\eta}{T_\nu B + \eta/3}}}{\eta^2}e^{-\frac{\eta m}{T_\nu^2 B^2 + T_\nu B\eta/3}},$$

where $\mathcal{P}(D, m)$ is defined in (20). Therefore (19) holds with:

$$c^* = \frac{2(T_\nu^2 B^2 + T_\nu B\eta/3)e^{\frac{2\eta}{T_\nu B + \eta/3}}}{\eta^2},$$
$$\beta^* = \frac{\eta}{T_\nu^2 B^2 + T_\nu B\eta/3}. \quad (34)$$

It is easy to see that c^* and β^* are both independent of V. ■

Note from (30) and (31) that Theorem 1 indeed holds for any finite $\boldsymbol{q}(0)$. We will later use this fact to prove the performance of FQLA. The following theorem is a special case of Theorem 1 and gives a more direct illustration of Theorem 1. Recall that $\mathcal{P}(D, m)$ is defined in (20). Define:

$$\mathcal{P}^{(r)}(D, m) \quad (35)$$
$$\triangleq \limsup_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}Pr\{\exists\, j, |q_j(\tau) - \gamma_{Vj}^*| > D + m\}.$$

***Theorem** 2:* If the condition in Theorem 1 holds and $S(t)$ is i.i.d., then under QLA, for any $c > 0$:

$$\mathcal{P}(D_1, cK_1\log(V)) \leq \frac{c_1^*}{V^c}, \quad (36)$$
$$\mathcal{P}^{(r)}(D_1, cK_1\log(V)) \leq \frac{c_1^*}{V^c}, \quad (37)$$

where $D_1 = \frac{2B^2}{L} + \frac{L}{4}$, $K_1 = \frac{B^2+BL/6}{L/2}$ and $c_1^* = \frac{8(B^2+BL/6)e^{\frac{L}{B+L/6}}}{L^2}$.

Proof: First we note that when $S(t)$ is i.i.d., we have $T_\nu = 1$ for $\nu = 0$. Now choose $\nu = 0$, $T_\nu = 1$ and $\eta = L/2$, then we see from (18) that

$$D = \max\left[\frac{2B^2 - L^2/4}{L}, \frac{L}{2}\right] \leq \frac{2B^2}{L} + \frac{L}{4}.$$

Now by (34) we see that (19) holds with $c^* = c_1^*$ and $\beta^* = \frac{L/2}{B^2+BL/6}$. Thus by taking $D_1 = \frac{2B^2}{L} + \frac{L}{4}$, we have:

$$\mathcal{P}(D_1, cK_1\log(V)) \quad \leq \quad c^* e^{-cK_1\beta^*\log(V)} = c_1^* e^{-c\log(V)},$$

where the last step follows since $\beta^* K_1 = 1$. Thus (36) follows. Equation (37) follows from (36) by using the fact that for any constant ζ, the events $\mathcal{E}_1 = \{\exists\, j, |q_j(\tau) - \gamma^*_{Vj}| > \zeta\}$ and $\mathcal{E}_2 = \{\|\boldsymbol{q}(\tau) - \boldsymbol{\gamma}^*_V\| > \zeta\}$ satisfy $\mathcal{E}_1 \subset \mathcal{E}_2$. Thus: $Pr\{\exists\, j, |q_j(\tau) - \gamma^*_{Vj}| > \zeta\} \leq Pr\{\|\boldsymbol{q}(\tau) - \boldsymbol{\gamma}^*_V\| > \zeta\}$. ■

Theorem 2 can be viewed as showing that for a large V, the probability for $q_j(t)$ to deviate from the j^{th} component of $\boldsymbol{\gamma}^*_V$ is exponentially decreasing in the distance. Thus it rarely deviates from γ^*_{Vj} by more than $\Theta(\log(V))$ distance.

B. When $g_0()$ is "locally smooth"

In this section, we consider the backlog behavior under QLA, for the case where the dual function $g_0(\boldsymbol{\gamma})$ is "locally smooth" at $\boldsymbol{\gamma}^*_0$. Specifically, we say that the function $g_0(\boldsymbol{\gamma})$ is *locally smooth* at $\boldsymbol{\gamma}^*_0$ with parameters $\varepsilon, L > 0$ if for all $\boldsymbol{\gamma} \succeq \boldsymbol{0}$ such that $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^*_0\| < \varepsilon$, we have:

$$g_0(\boldsymbol{\gamma}^*_0) \geq g_0(\boldsymbol{\gamma}) + L\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^*_0\|^2, \tag{38}$$

This condition contains the case when $g_0(\boldsymbol{\gamma})$ is twice differentiable with $\nabla g(\boldsymbol{\gamma}^*_0) = \boldsymbol{0}$ and $\boldsymbol{a}^T\nabla^2 g(\boldsymbol{\gamma})\boldsymbol{a} \leq -2L\|\boldsymbol{a}\|^2, \forall \boldsymbol{a}$ for any $\boldsymbol{\gamma}$ with $\|\boldsymbol{\gamma}^*_0 - \boldsymbol{\gamma}\| < \varepsilon$. Such a case usually occurs when the sets $\mathcal{X}^{(s_i)}, i = 1, ..., M$ are convex, thus a "continuous" set of actions are available. Notice that (38) is a looser condition than (15) in the neighborhood of $\boldsymbol{\gamma}^*_0$. As we will see, such structural difference of $g_0(\boldsymbol{\gamma})$ in the neighborhood of $\boldsymbol{\gamma}^*_0$ greatly affects the behavior of backlogs under QLA.

Theorem 3: If $g_0(\boldsymbol{\gamma})$ is locally smooth at $\boldsymbol{\gamma}^*_0$ with parameters $\varepsilon, L > 0$, independent of V, then under QLA with a sufficiently large V, we have:

(a) There exists $D = \Theta(\sqrt{V})$ such that whenever $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| \geq D$, we have:

$$\mathbb{E}\big\{\|\boldsymbol{q}(t+1) - \boldsymbol{\gamma}^*_V\| \mid \boldsymbol{q}(t)\big\} \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| - \frac{1}{\sqrt{V}}. \tag{39}$$

(b) $\mathcal{P}(D, m) \leq c^* e^{-\beta^* m}$, where $\mathcal{P}(D, m)$ is defined in (20), $c^* = \Theta(V)$ and $\beta^* = \Theta(1/\sqrt{V})$.

Theorem 3 can be viewed as showing that, when $g_0(\boldsymbol{\gamma})$ is locally smooth at $\boldsymbol{\gamma}^*_0$, the backlog vector will mostly be within $O(\sqrt{V}\log(V))$ distance from $\boldsymbol{\gamma}^*_V$. This contrasts with Theorem 1, which shows that the backlog will mostly be within $O(\log(V))$ distance from $\boldsymbol{\gamma}^*_V$. Intuitively, this is due to the fact that under local smoothness, the drift towards $\boldsymbol{\gamma}^*_V$ is smaller as $\boldsymbol{q}(t)$ gets closer to $\boldsymbol{\gamma}^*_V$, hence a $\Theta(\sqrt{V})$ distance is needed to guarantee a drift of size $\Theta(1/\sqrt{V})$; whereas under (15), any nonzero $\Theta(1)$ deviation from $\boldsymbol{\gamma}^*_V$ roughly generates a drift of size $\Theta(1)$ towards $\boldsymbol{\gamma}^*_V$, ensuring the backlog stays within $O(\log(V))$ distance from $\boldsymbol{\gamma}^*_V$. To prove Theorem 3, we need the following corollary of Lemma 2.

Corollary 1: If $S(t)$ is i.i.d., then under QLA,

$$\mathbb{E}\big\{\|\boldsymbol{q}(t+1) - \boldsymbol{\gamma}^*_V\|^2 \mid \boldsymbol{q}(t)\big\} \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\|^2 + 2B^2 \\ -2\big(g(\boldsymbol{\gamma}^*_V) - g(\boldsymbol{q}(t))\big).$$

Proof: When $S(t)$ is i.i.d., we have $T_\nu = 1$ for $\nu = 0$. ■

Proof: (Theorem 3) Part (a): We first see that for any $\boldsymbol{\gamma}$ with $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^*_V\| < \varepsilon V$, we have $\|\boldsymbol{\gamma}/V - \boldsymbol{\gamma}^*_0\| < \varepsilon$. Therefore,

$$g_0(\boldsymbol{\gamma}^*_0) \geq g_0(\boldsymbol{\gamma}/V) + L\|\boldsymbol{\gamma}/V - \boldsymbol{\gamma}^*_0\|^2. \tag{40}$$

Multiply both sides with V, we get:

$$g(\boldsymbol{\gamma}^*_V) \geq g(\boldsymbol{\gamma}) + \frac{L}{V}\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^*_V\|^2. \tag{41}$$

Similar as in the proof of Theorem 1 and by Corollary 1, we see that for (39) to hold, we need $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| \geq \frac{1}{\sqrt{V}}$ and:

$$2B^2 - 2\big(g(\boldsymbol{\gamma}^*_V) - g(\boldsymbol{q}(t))\big) \leq \frac{1}{V} - \frac{2}{\sqrt{V}}\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\|,$$

which can be rewritten as:

$$g(\boldsymbol{\gamma}^*_V) \geq g(\boldsymbol{q}(t))\big) + \frac{1}{\sqrt{V}}\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| + \frac{2B^2 - \frac{1}{V}}{2}. \tag{42}$$

By (41), we see that for (42) to hold, we only need:

$$\frac{L}{V}\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\|^2 \geq \frac{1}{\sqrt{V}}\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| + B^2. \tag{43}$$

It is easy to see that (43) holds whenever:

$$\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| \geq \frac{\frac{1}{\sqrt{V}} + \sqrt{\frac{1}{V} + \frac{4B^2L}{V}}}{2L/V} = \frac{\sqrt{V} + \sqrt{V + 4B^2LV}}{2L}$$

Denote $D = \frac{\sqrt{V}+\sqrt{V+4B^2LV}}{2L}$. We see now when V is large, (39) holds for any $\boldsymbol{q}(t)$ with $D \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| < \varepsilon V$. Now since $g(\boldsymbol{\gamma})$ is concave, it is easy to show that (42) holds for all $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| \geq D$. Hence (39) holds for all $\|\boldsymbol{q}(t) - \boldsymbol{\gamma}^*_V\| \geq D$, proving Part (a).

Part (b): By an argument that is similar as in the proof of Theorem 1, we see that Part (b) follows with: $\beta^* = \frac{3}{3\sqrt{V}B^2+B}$ and $c^* = 2(VB^2 + B\sqrt{V}/3)e^{\frac{6}{3B\sqrt{V}+1}}$. ■

C. Discussion of the choices of $g_0(\boldsymbol{\gamma})$

Note that in our analysis, we have focused only on the dual function $g_0(\boldsymbol{\gamma})$ being either locally polyhedral or locally smooth. These choices are made based on their practical generality. To be more precise, assume without loss of generality that there is only one network state and the set of feasible actions is a compact subset of $\mathbb{R}^n$. In practice, this action set is usually finite due to digitization. Thus we see from the definition of $g_0(\boldsymbol{\gamma})$ that an action, if chosen given a Lagrange multiplier $\boldsymbol{\gamma}$, will remain the chosen action for a range of Lagrange multipliers around $\boldsymbol{\gamma}$. Hence $g_0(\boldsymbol{\gamma})$ is polyhedral in this case. Now if the granularity of the action sets becomes finer and finer, we can expect the dual function $g_0(\boldsymbol{\gamma})$ to be "smoother and smoother," in the sense that moving from one action to another close-by action does not affect the value of

$g_0(\boldsymbol{\gamma})$ by much. Eventually if the granularity is fine enough then the action set can be viewed as convex. Now if the optimal network performance is achieved at some action not at the boundary of the action set, then we see that in a small neighborhood of $\boldsymbol{\gamma}_V^*$, we will usually have a locally smooth $g_0(\boldsymbol{\gamma})$ function. Further note that in both cases, the structure of $g_0(\boldsymbol{\gamma})$ is independent of V. Hence the conditions in Theorem 1 and 3 can typically be satisfied in practice.

D. Implications of Theorem 1 and 3

Consider the following simple problem: an operator operates a single queue and tries to support a Bernoulli arrival, i.e., either 1 or 0 packet arrives every slot, with rate $\lambda = 0.5$ (the rate may be unknown to the operator) with minimum energy expenditure. The channel is time-invariant. The rate-power curve over the channel is given by: $\mu(t) = \log(1 + P(t))$, where $P(t)$ is the allocated power at time t. Thus to obtain a rate of $\mu(t)$, we need $P(t) = e^{\mu(t)} - 1$. In every time slot, the operator decides how much power to allocate and serves the queue at the corresponding rate, with the goal of minimizing the time average power consumption subject to queue stability. Let Φ denote the time average energy expenditure incurred by the optimal policy. It is not difficult to see that $\Phi = e^{0.5} - 1$.

Now we look at the deterministic problem:

$$\min: \ V(e^{\mu} - 1), \qquad s.t.: \ 0.5 \leq \mu$$

It is easy to obtain $g(\gamma) = \inf_{\mu} \big\{V(e^{\mu} - 1) + \gamma(0.5 - \mu)\big\}$. Hence by the KKT conditions [16] one obtains that $\gamma_V^* = Ve^{0.5}$ and the optimal policy is to serve the queue at the constant rate $\mu^* = 0.5$. Suppose now QLA is applied to the problem. Then at slot t, if $q(t) = q$, QLA chooses the power to achieve the rate $\mu(t)$ such that ($[a]^+ = \max[a, 0]$):

$$\mu(t) \in \arg\min\{V(e^{\mu} - 1) + q(0.5 - \mu)\} = \big[\log(\frac{q}{V})\big]^+. \quad (44)$$

which incurs an instantanous power consumption of $P(t) \approx \frac{q(t)}{V} - 1$. In this case, it can be shown that Theorem 3 applies. Thus for most of the time $q(t) \in [\gamma_V^* - \sqrt{V}, \gamma_V^* + \sqrt{V}]$, i.e., $q(t) \in [Ve^{0.5} - \sqrt{V}, Ve^{0.5} + \sqrt{V}]$. Hence it is almost always the case that: $\log(e^{0.5} - \frac{1}{\sqrt{V}}) \leq \mu(t) \leq \log(e^{0.5} + \frac{1}{\sqrt{V}})$, which implies: $0.5 - \frac{1}{\sqrt{V}} \leq \mu(t) \leq 0.5 + \frac{1}{\sqrt{V}}$. Thus by a similar argument as in [8], one can show that $\overline{P} \leq \Phi + O(1/V)$, where $\overline{P}$ is the average power consumption.

Now consider the case when we can only choose to operate at $\mu \in \{0, \frac{1}{4}, \frac{3}{4}, 1\}$, with the corresponding power consumptions being: $P \in \{0, e^{\frac{1}{4}} - 1, e^{\frac{3}{4}} - 1, e - 1\}$. One can similarly obtain $\Phi = \frac{1}{2}(e^{\frac{3}{4}} + e^{\frac{1}{4}})$ and $\gamma_V^* = 2V(e^{\frac{3}{4}} - e^{\frac{1}{4}})$. In this case, Φ is achieved by time sharing the two rates $\{\frac{1}{4}, \frac{3}{4}\}$ with equal portion of time. It can also be shown that Theorem 1 applies in this case. Thus we see that under QLA, $q(t)$ is mostly within $\log(V)$ distance to γ_V^*. Hence by (44), we see that QLA almost always chooses between the two rates $\{\frac{1}{4}, \frac{3}{4}\}$, and uses them with almost equal frequencies. Hence QLA is also able to achieve $\overline{P} = \Phi + O(1/V)$ in this case.

The above argument can be generalized to many stochastic network optimization problems. Thus, we see that Theorem 1 and 3 not only provide us with probabilistic deviation bounds of $\boldsymbol{q}(t)$ from $\boldsymbol{\gamma}_V^*$, but also help to explain why QLA is able to achieve the desired utility performance: *under QLA, $\boldsymbol{q}(t)$ always stays close to $\boldsymbol{\gamma}_V^*$, hence the chosen action is always close to the set of optimal actions.*

E. Discussion of Scalability of Theorem 1 and 3

We note that though our results hold for many general multi-hop networks, the decaying exponents in Theorem 1 and 3 will usually depend on the network size r, e.g., B is a function of r. Hence the attraction may be looser as the network size r increases. Verifying whether the exponents in Theorem 1 and 3 are optimal with respect to r will be an interesting future research topic.

VI. The FQLA Algorithm

In this section, we propose a family of *Fast Quadratic Lyapunov based Algorithms* (FQLA) for general stochastic network optimization problems. We first provide an example to illustrate the idea of FQLA. We then describe FQLA with known $\boldsymbol{\gamma}_V^*$, called FQLA-Ideal, and study its performance. After that, we describe the more general FQLA without such knowledge, called FQLA-General. For brevity, we only describe FQLA for the case when $g_0(\boldsymbol{\gamma})$ is locally polyhedral. FQLA for the other case is discussed in [18].

A. FQLA: a Single Queue Example

To illustrate the idea of FQLA, we first look at an example. Figure 3 shows a 10^4-slot sample backlog process under QLA.[3] We see that after roughly 1500 slots, $q(t)$ always stays very close to γ_V^*, which is a $\Theta(V)$ scalar in this case. To reduce delay, we can first find $\mathcal{W} \in (0, \gamma_V^*)$ such that: under QLA, there exists a time t_0 so that $q(t_0) \geq \mathcal{W}$ and once $q(t) \geq \mathcal{W}$, it remains so for all time (the solid line in Fig. 3 shows one for these 10^4 slots). We then place $\mathcal{W}$ fake bits (called *place-holder bits* [11]) in the queue at time 0, i.e., initialize $q(0) = \mathcal{W}$, and run QLA. It is easy to show that the utility performance of QLA will remain the same with this change, and the average backlog is now reduced by $\mathcal{W}$. However, such a $\mathcal{W}$ may require $\mathcal{W} = \gamma_V^* - \Theta(V)$, thus the average backlog may still be $\Theta(V)$.

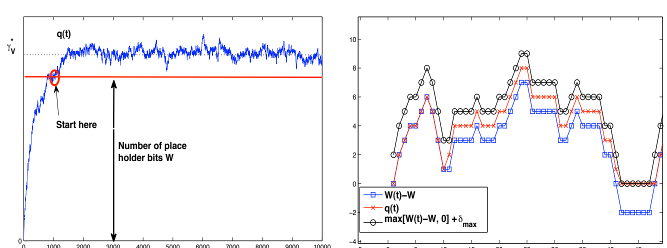

Fig. 3. Left: A sample backlog process; Right: Example of $W(t)$ and $q(t)$.

FQLA instead finds a $\mathcal{W}$ such that in steady state, the backlog process under QLA *rarely* goes below it, and places $\mathcal{W}$ place-holder bits in the queue at time 0. FQLA then uses an auxiliary process $W(t)$, called the *virtual backlog process*, to keep track of the backlog process that should have

[3] This sample backlog process is one sample backlog process of queue 1 of the system considered in Section VII, under QLA with $V = 50$.

been generated if QLA is used. Specifically, FQLA initializes $W(0) = \mathcal{W}$. Then at every slot, QLA is run using $W(t)$ as the queue size, and $W(t)$ is updated according to QLA. With $W(t)$ and $\mathcal{W}$, FQLA works as follows: At time t, if $W(t) \geq \mathcal{W}$, FQLA performs QLA's action (obtained based on $S(t)$ and $W(t)$); else if $W(t) < \mathcal{W}$, FQLA carefully modifies QLA's action so as to maintain $q(t) \approx \max[W(t) - \mathcal{W}, 0]$ for all t (see Fig.3 for an example). Similar as above, this roughly reduces the average backlog by $\mathcal{W}$. The difference is that now we can show that $\mathcal{W} = \max[\gamma_V^* - [\log(V)]^2, 0]$ meets the requirement. Thus it is possible to bring the average backlog down to $O([\log(V)]^2)$. Also, since $W(t)$ can be viewed as a backlog process generated by QLA, it rarely goes below $\mathcal{W}$ in steady state. Hence FQLA is almost always the same as QLA, thus is able to achieve an $O(1/V)$ close-to-optimal utility performance.

B. The FQLA-Ideal Algorithm

In this section, we present the FQLA-Ideal algorithm. We assume the value $\boldsymbol{\gamma}_V^* = (\gamma_{V1}^*, ..., \gamma_{Vr}^*)^T$ is known a-priori.

FQLA-Ideal:

(I) *Determining place-holder bits:* For each j, define:

$$\mathcal{W}_j = \max\left[\gamma_{Vj}^* - [\log(V)]^2, 0\right], \tag{45}$$

as the number of *place-holder bits* of queue j.

(II) *Place-holder-bit based action:* Initialize $q_j(0) = 0$, and $W_j(0) = \mathcal{W}_j,\ \forall j$. For $t \geq 1$, observe the network state $S(t)$, solve (7) with $\boldsymbol{W}(t)$ in place of $\boldsymbol{q}(t)$. Perform the chosen action with the following modification: Let $\boldsymbol{A}(t)$ and $\boldsymbol{\mu}(t)$ be the arrival and service rate vectors generated by the action. For each queue j, do (Idle fill if needed):

a) If $W_j(t) \geq \mathcal{W}_j$: admit $A_j(t)$ arrivals, serve $\mu_j(t)$ data, i.e., update the backlog by:

$$q_j(t+1) = \max\left[q_j(t) - \mu_j(t), 0\right] + A_j(t).$$

b) If $W_j(t) < \mathcal{W}_j$: admit $\tilde{A}_j(t) = \max\left[A_j(t) - \mathcal{W}_j + W_j(t), 0\right]$ arrivals, serve $\mu_j(t)$ data, i.e., update the backlog by:

$$q_j(t+1) = \max\left[q_j(t) - \mu_j(t), 0\right] + \tilde{A}_j(t).$$

c) Update $W_j(t)$ by:

$$W_j(t+1) = \max\left[W_j(t) - \mu_j(t), 0\right] + A_j(t).$$

From above we see that FQLA-Ideal is the same as QLA based on $\boldsymbol{W}(t)$ when $W_j(t) \geq \mathcal{W}_j$ for all j. When $W_j(t) < \mathcal{W}_j$ for some queue j, FQLA-Ideal admits roughly the *excessive* packets after $W_j(t)$ is brought back to be above $\mathcal{W}_j$ for the queue. Thus for problems where QLA admits an easy implementation, e.g., [3], [5], it is also easy to implement FQLA. However, we also notice two different features of FQLA: (1) By (45), $\mathcal{W}_j$ can be 0. However, when V is large, this happens only when $\gamma_{0j}^* = \gamma_{Vj}^* = 0$ according to Lemma 1. In this case $\mathcal{W}_j = \gamma_{Vj}^* = 0$, and queue j indeed needs zero place-holder bits. (2) Packets may be dropped in Step II-(b) upon their arrivals, or after they are admitted into the network in a multihop problem. Such packet dropping is natural in many flow control problems and does not change the nature of these problems. In other problems where such option is not available, the packet dropping option is introduced to achieve desired delay performance, and it can be shown that the fraction of packets dropped can be made arbitrarily small. Note that packet dropping here is to compensate for the deviation from the desired Lagrange multiplier, thus is different from that in [19], where packet dropping is used for drift steering.

C. Performance of FQLA-Ideal

Here we look at the performance of FQLA-Ideal. We first have the following lemma that shows the relationship between $\boldsymbol{q}(t)$ and $\boldsymbol{W}(t)$ under FQLA-Ideal. We will use it later to prove the delay bound of FQLA. Note that the lemma also holds for FQLA-General described later, as FQLA-Ideal/General differ only in the way of determining $\boldsymbol{\mathcal{W}} = (\mathcal{W}_1, ..., \mathcal{W}_r)^T$.

***Lemma** 3:* Under FQLA-Ideal/General, we have $\forall\ j, t$:

$$\max\left[W_j(t) - \mathcal{W}_j, 0\right] \leq q_j(t) \leq \max\left[W_j(t) - \mathcal{W}_j, 0\right] + \delta_{max} \tag{46}$$

where δ_{max} is defined in Section III-B to be the upper bound of the number of arriving or departing packets of a queue.

Proof: See Appendix C. ■

The following theorem summarizes the main performance results of FQLA-Ideal. Recall that for a given policy π, f_{av}^π denotes its average cost defined in (5) and $f^\pi(t)$ denotes the cost induced by π at time t.

***Theorem** 4:* If the condition in Theorem 1 holds and a steady state distribution exists for the backlog process generated by QLA, then with a sufficiently large V, we have under FQLA-Ideal that,

$$\overline{q} = O([\log(V)]^2), \tag{47}$$

$$f_{av}^{FI} = f_{av}^* + O(1/V), \tag{48}$$

$$P_{drop} = O(1/V^{c_0 \log(V)}), \tag{49}$$

where $c_0 = \Theta(1)$, $\overline{q}$ is the time average backlog, f_{av}^{FI} is the expected time average cost of FQLA-Ideal, f_{av}^* is the optimal time average cost and P_{drop} is the time average fraction of packets that are dropped in Step-II (b).

Proof: Since a steady state distribution exists for the backlog process generated by QLA, we see that $\mathcal{P}(D, m)$ in (20) represents the steady state probability of the event that the backlog vector deviates from $\boldsymbol{\gamma}_V^*$ by distance $D+m$. Now since $\boldsymbol{W}(t)$ can be viewed as a backlog process generated by QLA, with $\boldsymbol{W}(0) = \boldsymbol{\mathcal{W}}$ instead of $\boldsymbol{0}$, we see from the proof of Theorem 1 that Theorem 1 and 2 hold for $\boldsymbol{W}(t)$, and by [7], QLA based on $\boldsymbol{W}(t)$ achieves an average cost of $f_{av}^* + O(1/V)$. Hence by Theorem 2, there exist constants $D_1, K_1, c_1^* = \Theta(1)$ so that: $\mathcal{P}^{(r)}(D_1, cK_1\log(V)) \leq \frac{c_1^*}{V^c}$. By the definition of $\mathcal{P}^{(r)}(D_1, cK_1\log(V))$, this implies that in steady state: $Pr\{W_j(t) > \gamma_{Vj}^* + D_1 + m\} \leq c_1^* e^{-\frac{m}{K_1}}$.

Now let: $Q_j(t) = \max[W_j(t) - \gamma_{Vj}^* - D_1, 0]$. We see that $Pr\{Q_j(t) > m\} \leq c_1^* e^{-\frac{m}{K_1}},\ \forall m \geq 0$. We thus have $\overline{Q_j} = O(1)$, where $\overline{Q_j}$ is the time average value of $Q_j(t)$. Now it is easy to see by (45) and (46) that $q_j(t) \leq Q_j(t) + [\log(V)]^2 + D_1 + \delta_{max}$ for all t. Thus (47) follows since for a large V:

$$\overline{q_j} \leq \overline{Q_j} + [\log(V)]^2 + D_1 + \delta_{max} = \Theta([\log(V)]^2).$$

Now consider the average cost. To save space, we use FI for FQLA-Ideal. From above, we see that QLA based on $\boldsymbol{W}(t)$ achieves an expected average cost of $f_{av}^* + O(1/V)$. Thus it suffices to show that FQLA-Ideal performs almost the same as QLA based on $\boldsymbol{W}(t)$. First we have for all $t \geq 1$ that:

$$\frac{1}{t}\sum_{\tau=0}^{t-1} f^{FI}(\tau) = \frac{1}{t}\sum_{\tau=0}^{t-1} f^{FI}(\tau)1_{E(\tau)} + \frac{1}{t}\sum_{\tau=0}^{t-1} f^{FI}(\tau)1_{E^c(\tau)}.$$

Here $1_{E(\tau)}$ is the indicator function of the event $E(\tau)$, $E(\tau)$ is the event that FQLA-Ideal performs the same action as QLA at time τ, and $1_{E^c(\tau)} = 1 - 1_{E(\tau)}$. Taking expectation on both sides and using the fact that when FQLA-Ideal takes the same action as QLA, $f^{FI}(\tau) = f^{QLA}(\tau)$, we have:

$$\begin{aligned}\frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}\big\{f^{FI}(\tau)\big\} &\leq \frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}\big\{f^{QLA}(\tau)1_{E(\tau)}\big\} \\ &\quad + \frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}\big\{\delta_{max}1_{E^c(\tau)}\big\}.\end{aligned}$$

Taking the limit as t goes to infinity on both sides and using $f^{QLA}(\tau)1_{E(\tau)} \leq f^{QLA}(\tau)$, we get:

$$\begin{aligned} f_{av}^{FI} &\leq f_{av}^{QLA} + \delta_{max}\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}\big\{1_{E^c(\tau)}\big\} \\ &= f_{av}^{QLA} + \delta_{max}\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}Pr\{E^c(\tau)\}. \quad (50)\end{aligned}$$

However, $E^c(\tau)$ is included in the event that there exists a j such that $W_j(\tau) < \mathcal{W}_j$. Therefore by (37) in Theorem 2, for a large V such that $\frac{1}{2}[\log(V)]^2 \geq D_1$ and $\log(V) \geq 8K_1$,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}Pr\{E^c(\tau)\} &\leq \mathcal{P}^{(r)}(D_1, [\log(V)]^2 - D_1) \\ &= O(c_1^*/V^{\frac{1}{2K_1}\log(V)}) \\ &= O(1/V^4). \quad (51)\end{aligned}$$

Using this fact in (50), we obtain:

$$f_{av}^{FI} = f_{av}^{QLA} + O(\delta_{max}/V^4) = f_{av}^* + O(1/V),$$

where the last equality holds since $f_{av}^{QLA} = f_{av}^* + O(1/V)$. This proves (48). (49) follows since packets are dropped at time τ only if $E^c(\tau)$ happens, thus by (51), the fraction of time when packet dropping happens is $O(1/V^{c_0\log(V)})$ with $c_0 = \frac{1}{2K_1} = \Theta(1)$, and each time no more than $\sqrt{r}B$ packets can be dropped. ■

D. The FQLA-General algorithm

Now we describe the FQLA algorithm without any a-priori knowledge of $\boldsymbol{\gamma}_V^*$, called FQLA-General. FQLA-General first runs the system for a long enough time T, such that the system enters its steady state. Then it chooses a sample of the queue vector value to estimate $\boldsymbol{\gamma}_V^*$ and uses that to decide $\boldsymbol{\mathcal{W}}$.

<u>FQLA-General:</u>

(I) <u>Determining place-holder bits:</u>

a) Choose a large time T (See Section VI-E for the size of T) and initialize $\boldsymbol{W}(0) = \boldsymbol{0}$. Run the QLA algorithm with parameter V, at every time slot t, update $\boldsymbol{W}(t)$ according to the QLA algorithm and obtain $\boldsymbol{W}(T)$.

b) For each queue j, define:

$$\mathcal{W}_j = \max\big[W_j(T) - [\log(V)]^2, 0\big], \quad (52)$$

as the number of *place-holder bits*.

(II) <u>Place-holder-bit based action:</u> same as FQLA-Ideal.

The performance of FQLA-General is summarized as follows:

***Theorem** 5:* Assume the conditions in Theorem 4 hold and the system is in steady state at time T, then under FQLA-General with a sufficiently large V, with probability $1 - O(\frac{1}{V^4})$: (a) $\overline{q} = O([\log(V)]^2)$, (b) $f_{av}^{FG} = f_{av}^* + O(1/V)$, and (c) $P_{drop} = O(1/V^{c_0\log(V)})$, where $c_0 = \Theta(1)$ and f_{av}^{FG} is the expected time average cost of FQLA-General.

Proof: We will show that with probability of $1 - O(\frac{1}{V^4})$, $\mathcal{W}_j$ is close to $\max[\gamma_{Vj}^* - [\log(V)]^2, 0]$. The rest can then be proven similarly as in the proof of Theorem 4.

For each queue j, define:

$$v_j^+ = \gamma_{Vj}^* + \frac{1}{2}[\log(V)]^2, \quad v_j^- = \max\big[\gamma_{Vj}^* - \frac{1}{2}[\log(V)]^2, 0\big].$$

Note that v_j^- is defined with a $\max[\,]$ operator. This is due to the fact that γ_{Vj}^* can be zero. As in (51), we see that by Theorem 2, there exists $D_1 = \Theta(1), K_1 = \Theta(1)$ such that if V is such that $\frac{1}{4}\log^2(V) \geq D_1$ and $\log(V) \geq 16K_1$, then:

$$\begin{aligned} Pr\big\{\exists\, j,\, W_j(T) \notin [v_j^-, v_j^+]\big\} &\leq \mathcal{P}^{(r)}(D_1, \frac{1}{2}[\log(V)]^2 - D_1) \\ &= O(1/V^4).\end{aligned}$$

Thus $Pr\big\{W_j(T) \in [v_j^-, v_j^+]\,\forall j\big\} = 1 - O(1/V^4)$, implying:

$$Pr\big\{\mathcal{W}_j \in [\hat{v}_j^-, \hat{v}_j^+] \quad \forall j\big\} = 1 - O(1/V^4).$$

where $\hat{v}_j^+ = \max\big[\gamma_{Vj}^* - \frac{1}{2}[\log(V)]^2, 0\big]$ and $\hat{v}_j^- = \max\big[\gamma_{Vj}^* - \frac{3}{2}[\log(V)]^2, 0\big]$. Hence for a large V, with probability $1 - O(\frac{1}{V^4})$: if $\gamma_{Vj}^* > 0$, we have $\gamma_{Vj}^* - \frac{3}{2}[\log(V)]^2 \leq \mathcal{W}_j \leq \gamma_{Vj}^* - \frac{1}{2}[\log(V)]^2$; else if $\gamma_{Vj}^* = 0$, we have $\mathcal{W}_j = \gamma_{Vj}^*$. The rest of the proof is similar to the proof of Theorem 4. ■

E. Practical Issues

From Lemma 1 we see that the magnitude of $\boldsymbol{\gamma}_V^*$ can be $\Theta(V)$. This means that T in FQLA-General may need to be $\Omega(V)$, which is not very desirable when V is large. We can instead use the following heuristic method to accelerate the process of determining $\boldsymbol{\mathcal{W}}$: For every queue j, guess a very large $\mathcal{W}_j$. Then start with this $\boldsymbol{\mathcal{W}}$ and run the QLA algorithm for some T_1, say $\sqrt{V}$ slots. Observe the resulting backlog process. Modify the guess for each queue j using a bisection algorithm until a proper $\boldsymbol{\mathcal{W}}$ is found, i.e. when running QLA from $\boldsymbol{\mathcal{W}}$, we observe fluctuations of $W_j(t)$ around $\mathcal{W}_j$ instead of a nearly constant increase or decrease for all j. Then let $\mathcal{W}_j = \max[\mathcal{W}_j - [\log(V)]^2, 0]$. To further reduce the error probability, one can repeat Step-I (a) multiple times and use the average value as $\boldsymbol{W}(T)$.

VII. SIMULATION

In this section we provide simulation results for the FQLA algorithms. For simplicity, we only consider the case where $g_0(\gamma)$ is locally polyhedral. We consider a five queue system similar to the example in Section III-D. In this case $r = 5$. The system is shown in Fig. 4. The goal is to perform power allocation at each node so as to support the arrival with minimum energy expenditure.

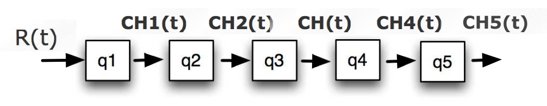

Fig. 4. A five queue system

In this example, the random network state $S(t)$ is the vector $(R(t), CH_i(t), i = 1, .., 5)$. Similar as in Section III-D, we have: $\boldsymbol{A}(t) = (R(t), \mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t))^T$ and $\boldsymbol{\mu}(t) = (\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t), \mu_5(t))^T$, i.e., $A_1(t) = R(t)$, $A_i(t) = \mu_{i-1}(t)$ for $i \geq 2$, where $\mu_i(t)$ is the service rate obtained by queue i at time t. $R(t)$ is 0 or 2 with probabilities $\frac{3}{8}$ and $\frac{5}{8}$, respectively. $CH_i(t)$ can be "Good" or "Bad" with equal probabilities for $1 \leq i \leq 5$. When the channel is good, one unit of power can serve two packets; otherwise it can serve only one. We assume $CH_i(t)$ are all independent and all channels can be activated at the same time without affecting others. It can be verified that $\boldsymbol{\gamma}_V^* = (5V, 4V, 3V, 2V, V)^T$ is unique. In this example, the backlog vector process evolves as a Markov chain with countably many states. Thus there exists a stationary distribution for the backlog vector under QLA.

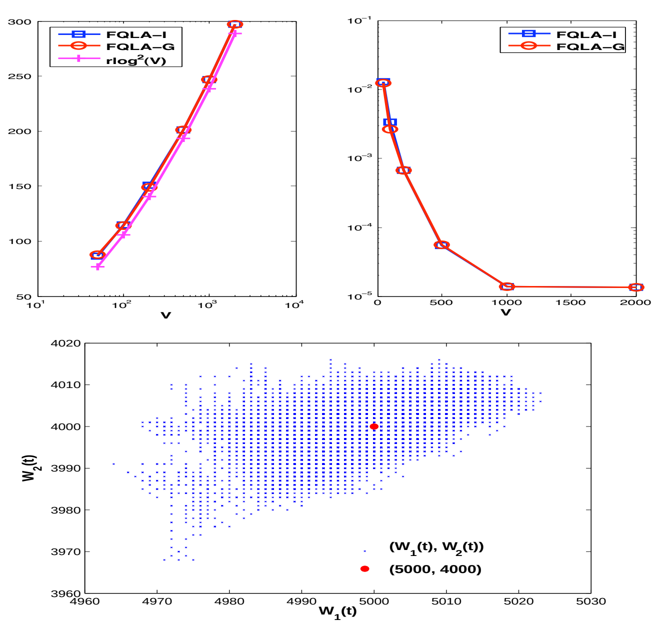

Fig. 5. FQLA-Ideal performance: Up-Left - Average queue size; Up-Right - Percentage of packets dropped; Bottom - Sample $(W_1(t), W_2(t))$ process for $t \in [10000, 110000]$ and $V = 1000$ under FQLA-Ideal.

We simulate FQLA-Ideal and FQLA-General with $V = 50, 100, 200, 500, 1000$ and 2000. We run each case for 5×10^6 slots. For FQLA-General, we use $T = 50V$ in Step-I and repeat Step-I 100 times and use their average as $\boldsymbol{W}(T)$. The up-left plot in Fig. 5 shows that the average queue sizes under both FQLAs are always close to the value $5[\log(V)]^2$ ($r = 5$). The up-right plot shows that the percentage of packets dropped decreases rapidly and gets below 10^{-4} when $V \geq 500$ under both FQLAs. These plots show that in practice, V may not have to be very large for Theorem 4 and 5 to hold. The bottom plot shows a sample $(W_1(t), W_2(t))$ process for a 10^5-slot interval under FQLA-Ideal with $V = 1000$, considering only the first two queues of Fig. 4. We see that $(W_1(t), W_2(t))$ always remains close to $(\gamma_{V1}^*, \gamma_{V2}^*) = (5V, 4V)$, and $W_1(t) \geq \mathcal{W}_1 = 4952$, $W_2(t) \geq \mathcal{W}_2 = 3952$. For all V values, the average power expenditure is very close to 3.75, which is the optimal energy expenditure, and the average of $\sum_j W_j(t)$ is very close to $15V$ (plots omitted for brevity).

Interestingly, the "attraction phenomenon" in the bottom plot of Fig. 5 was also observed in a recent paper [12], which implemented the QLA algorithm in a 40-node wireless sensor network testbed. It has also been shown in [12] that by using QLA plus LIFO, one can reduce the delay experienced by all but a small fraction of the network traffic by more than 90%. While this fact can not be explained by any previous results on QLA, it can easily be explained using Theorem 1 and 3 as follows: Consider a node j. Under LIFO, new packets entering Node j are placed to the front of the buffer. We also know that $q_j \in \mathcal{I} = [\gamma_{Vj}^* - [\log(V)]^2, \gamma_{Vj}^* + [\log(V)]^2]$ for most of the time. Thus most packets enter and leave Node j when $q_j \in \mathcal{I}$. Hence for most packets, Node j is a queue with on average no more than $2[\log(V)]^2$ packets. Hence most packets only need to wait for on average no more than $\Theta([\log(V)]^2)$ packets before getting served.

VIII. LAGRANGE MULTIPLIER: "SHADOW PRICE" AND "NETWORK GRAVITY"

[4] It is well known that Lagrange Multipliers can play the role of "shadow prices" to regulate flows in many flow-based problems with different objectives, e.g., [20]. This important feature has enabled the development of many distributed algorithms in resource allocation problems, e.g., [21]. However, a problem of this type typically requires data transmissions to be represented as flows. Thus in a network that is discrete in nature, e.g., time slotted or packetized transmission, a rate allocation solution obtained by solving such a flow-based problem does not immediately specify a scheduling policy.

Recently, several Lyapunov algorithms have been proposed to solve utility optimization problems under discrete network settings. In these algorithms, backlog vectors act as the "gravity" of the network and allow optimal scheduling to be built upon them. It is also revealed in [17] that QLA is closely related to the dual subgradient method and backlogs play the same role as Lagrange multipliers in a time invariant network. Now we see by Theorem 1 and 3 that the backlogs indeed play the same role as Lagrange multipliers even under a more general stochastic network.

[4]This section appeared in the WiOpt 2009 paper. However, it is not included in the IEEE TAC version due to space consideration.

In fact, the backlog process under QLA can be closely related to a sequence of updated Lagrange multipliers under a subgradient method. Consider the following randomized incremental subgradient method (RISM) [16], which makes use of the separable nature of (11) and solves the dual problem (10) as follows:

RISM: Initialize $\boldsymbol{\gamma}(0)$; at iteration t, observe $\boldsymbol{\gamma}(t)$, choose a random state $S(t)\in\mathcal{S}$ according to some probability law. (1) If $S(t)=s_i$, find $x^{(s_i)}_{\boldsymbol{\gamma}(t)}\in\mathcal{X}^{(s_i)}$ that solves the following:

$$\begin{aligned} \min: \quad & Vf(s_i,x)+\sum_j\gamma_j(t)\big[A_j(s_i,x)-\mu_j(s_i,x)\big] \\ s.t. \quad & x\in\mathcal{X}^{(s_i)}. \end{aligned} \tag{53}$$

(2) Using the $x^{(s_i)}_{\boldsymbol{\gamma}(t)}$ found, update $\boldsymbol{\gamma}(t)$ according to: [5]

$$\gamma_j(t+1)=\max\left[\gamma_j(t)-\alpha^t\mu_j(s_i,x^{(s_i)}_{\boldsymbol{\gamma}(t)}),0\right]+\alpha^tA_j(s_i,x^{(s_i)}_{\boldsymbol{\gamma}(t)}).$$

As an example, $S(t)$ can be chosen by independently choosing $S(t)=s_i$ with probability p_{s_i} every time slot. In this case $S(t)$ will be i.i.d.. Note that in the stochastic problem, a network state s_i is chosen randomly by nature as the physical system state at time t; while here a state is chosen artificially by RISM according some probability law. Now we see from (7) and (53) that: given $\boldsymbol{q}(t)=\boldsymbol{\gamma}(t)$ and s_i, *QLA and RISM choose an action in the same way*. If also $\alpha^t=1$ for all t, and that $S(t)$ under RISM evolves according to the same probability law as $S(t)$ of the physical system, we see that *applying QLA to the network is indeed equivalent to applying RISM to the dual problem of (9), with the network state being chosen by nature, and the network backlog being the Lagrange multiplier.* Therefore, Lagrange Multipliers under such stochastic discrete network settings act as the "network gravity," thus allow scheduling to be done optimally and adaptively based on them. This "network gravity" functionality of Lagrange Multipliers in discrete network problems can thus be viewed as the counterpart of their "shadow price" functionality in the flow-based problems. Further more, the "network gravity" property of Lagrange Multipliers enables the use of place holder bits to reduce network delay in network utility optimization problems. This is a unique feature not possessed by its "price" counterpart.

IX. Conclusion

In this paper, we study the backlog behavior under the class of QLA algorithms. We show that for every stochastic network optimization problem, the network backlog is "exponentially attracted" to an attractor, which is the dual optimal solution of a corresponding *deterministic* problem. Based on this finding, we develop the FQLA algorithm to achieve an $\big[O(1/V),O([\log(V)]^2)\big]$ performance-delay tradeoff for problems with a discrete set of action options, and a square-root tradeoff for continuous problems.

[5]Note that this update rule is different from RISM's usual rule, i.e., $\gamma_j(t+1)=\max\big[\gamma_j(t)-\alpha^t\mu_j(s_i,x)+\alpha^tA_j(s_i,x),0\big]$, but it almost does not affect the performance of RISM.

Appendix A- Proof of Lemma 2

Here we prove Lemma 2. First we prove the following:

Lemma 4: Under queueing dynamic (3), we have:

$$\begin{aligned} \|\boldsymbol{q}(t+1)-\boldsymbol{\gamma}_V^*\|^2 \quad \leq \quad & \|\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*\|^2+2B^2 \\ & -2\big(\boldsymbol{\gamma}_V^*-\boldsymbol{q}(t)\big)^T(\boldsymbol{A}(t)-\boldsymbol{\mu}(t)). \end{aligned}$$

Proof: (Lemma 4) From (3), we see that $\boldsymbol{q}(t+1)$ is obtained by first projecting $\boldsymbol{q}(t)-\boldsymbol{\mu}(t)$ onto $\mathbb{R}^r_+$ and then adding $\boldsymbol{A}(t)$. Thus we have (we use $[\boldsymbol{a}]^+$ to denote the projection of $\boldsymbol{a}$ onto $\mathbb{R}^r_+$):

$$\begin{aligned} & \|\boldsymbol{q}(t+1)-\boldsymbol{\gamma}_V^*\|^2 \\ = \; & \|[\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^++\boldsymbol{A}(t)-\boldsymbol{\gamma}_V^*\|^2 \\ = \; & \big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^++\boldsymbol{A}(t)-\boldsymbol{\gamma}_V^*\big)^T \\ & \qquad \big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^++\boldsymbol{A}(t)-\boldsymbol{\gamma}_V^*\big) \\ = \; & \big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^+-\boldsymbol{\gamma}_V^*\big)^T\big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^+-\boldsymbol{\gamma}_V^*\big) \\ & +2\big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^+-\boldsymbol{\gamma}_V^*\big)^T\boldsymbol{A}(t)+\|\boldsymbol{A}(t)\|^2. \end{aligned} \tag{54}$$

By the non expansive property of projection [16], we have:

$$\begin{aligned} & \big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^+-\boldsymbol{\gamma}_V^*\big)^T\big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^+-\boldsymbol{\gamma}_V^*\big) \\ \leq \; & \big(\boldsymbol{q}(t)-\boldsymbol{\mu}(t)-\boldsymbol{\gamma}_V^*\big)^T\big(\boldsymbol{q}(t)-\boldsymbol{\mu}(t)-\boldsymbol{\gamma}_V^*\big) \\ = \; & \|\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*\|^2+\|\boldsymbol{\mu}(t)\|^2-2(\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*)^T\boldsymbol{\mu}(t). \end{aligned}$$

Plug this into (54), we have:

$$\begin{aligned} & \|\boldsymbol{q}(t+1)-\boldsymbol{\gamma}_V^*\|^2 \\ \leq \; & \|\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*\|^2+\|\boldsymbol{\mu}(t)\|^2-2(\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*)^T\boldsymbol{\mu}(t) \\ & +\|\boldsymbol{A}(t)\|^2+2\big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^+-\boldsymbol{\gamma}_V^*\big)^T\boldsymbol{A}(t). \end{aligned} \tag{55}$$

Now since $\boldsymbol{q}(t),\boldsymbol{\mu}(t),\boldsymbol{A}(t)\succeq\boldsymbol{0}$, it is easy to see that:

$$\big([\boldsymbol{q}(t)-\boldsymbol{\mu}(t)]^+\big)^T\boldsymbol{A}(t)\leq\boldsymbol{q}(t)^T\boldsymbol{A}(t). \tag{56}$$

By (55) and (56) we have:

$$\begin{aligned} & \|\boldsymbol{q}(t+1)-\boldsymbol{\gamma}_V^*\|^2 \\ \leq \; & \|\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*\|^2+\|\boldsymbol{\mu}(t)\|^2-2(\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*)^T\boldsymbol{\mu}(t) \\ & +\|\boldsymbol{A}(t)\|^2+2\big(\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*\big)^T\boldsymbol{A}(t) \\ \leq \; & \|\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*\|^2+2B^2-2\big(\boldsymbol{\gamma}_V^*-\boldsymbol{q}(t)\big)^T(\boldsymbol{A}(t)-\boldsymbol{\mu}(t)), \end{aligned}$$

where the last inequality follows since $\|\boldsymbol{A}(t)\|^2\leq B^2$ and $\|\boldsymbol{\mu}(t)\|^2\leq B^2$. ■

We now prove Lemma 2.

Proof: (Lemma 2) By Lemma 4 we see that when $S(t)=s_i$, we have the following for any network state s_i with a given $\boldsymbol{q}(t)$ (here we add superscripts to $\boldsymbol{q}(t+1)$, $\boldsymbol{A}(t)$ and $\boldsymbol{\mu}(t)$ to indicate their dependence on s_i):

$$\begin{aligned} \|\boldsymbol{q}^{(s_i)}(t+1)-\boldsymbol{\gamma}_V^*\|^2 \leq \; & \|\boldsymbol{q}(t)-\boldsymbol{\gamma}_V^*\|^2+2B^2 \\ & -2(\boldsymbol{\gamma}_V^*-\boldsymbol{q}(t))^T(\boldsymbol{A}^{(s_i)}(t)-\boldsymbol{\mu}^{(s_i)}(t)). \end{aligned} \tag{57}$$

By definition, $A_j^{(s_i)}(t)=A_j(s_i,x^{(s_i)}_{\boldsymbol{q}(t)})$, and $\mu_j^{(s_i)}(t)=\mu_j(s_i,x^{(s_i)}_{\boldsymbol{q}(t)})$, with $x^{(s_i)}_{\boldsymbol{q}(t)}$ being the solution of (7) for the given

$\boldsymbol{q}(t)$. Now consider the deterministic problem (9) with only a single network state s_i, then the dual function (11) becomes:

$$g_{s_i}(\gamma) = \inf_{x^{(s_i)}\in\mathcal{X}^{(s_i)}} \bigg\{ Vf(s_i, x^{(s_i)}) + \sum_j \gamma_j \big[A_j(s_i, x^{(s_i)}) - \mu_j(s_i, x^{(s_i)})\big] \bigg\}. \quad (58)$$

Therefore by (12) we see that $(\boldsymbol{A}^{(s_i)}(t) - \boldsymbol{\mu}^{(s_i)}(t))$ is a subgradient of $g_{s_i}(\gamma)$ at $\boldsymbol{q}(t)$. Thus by (13) we have:

$$(\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t))^T(\boldsymbol{A}^{(s_i)}(t) - \boldsymbol{\mu}^{(s_i)}(t)) \geq g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t)). \quad (59)$$

Plug (59) into (57), we get:

$$\|\boldsymbol{q}^{(s_i)}(t+1) - \boldsymbol{\gamma}_V^*\|^2 \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 + 2B^2 - 2\big(g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big). \quad (60)$$

More generally, we have:

$$\|\boldsymbol{q}(t+1) - \boldsymbol{\gamma}_V^*\|^2 \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 + 2B^2 - 2\big(g_{S(t)}(\boldsymbol{\gamma}_V^*) - g_{S(t)}(\boldsymbol{q}(t))\big). \quad (61)$$

Now fix $\nu > 0$ and summing up (61) from time t to $t+T_\nu-1$,

$$\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 + 2T_\nu B^2 - 2\sum_{\tau=0}^{T_\nu-1}\big[g_{S(t+\tau)}(\boldsymbol{\gamma}_V^*) - g_{S(t+\tau)}(\boldsymbol{q}(t+\tau))\big]. \quad (62)$$

Adding and subtracting the term $2\sum_{\tau=0}^{T_\nu-1} g_{S(t+\tau)}(\boldsymbol{q}(t))$ from the right hand side, we obtain:

$$\begin{aligned}\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 &\leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 + 2T_\nu B^2 \\ &\quad -2\sum_{\tau=0}^{T_\nu-1}\big[g_{S(t+\tau)}(\boldsymbol{\gamma}_V^*) - g_{S(t+\tau)}(\boldsymbol{q}(t))\big] \\ &\quad +2\sum_{\tau=0}^{T_\nu-1}\big[g_{S(t+\tau)}(\boldsymbol{q}(t+\tau)) - g_{S(t+\tau)}(\boldsymbol{q}(t))\big]. \end{aligned} \quad (63)$$

Since $\|\boldsymbol{q}(t)-\boldsymbol{q}(t+\tau)\| \leq \tau B$ and $\|\boldsymbol{A}^{(s_i)}(t)-\boldsymbol{\mu}^{(s_i)}(t)\| \leq B$, using (59) and the fact that for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}^T\boldsymbol{b} \leq \|\boldsymbol{a}\|\|\boldsymbol{b}\|$, we have:

$$g_{S(t+\tau)}(\boldsymbol{q}(t+\tau)) - g_{S(t+\tau)}(\boldsymbol{q}(t)) \leq \tau B^2. \quad (64)$$

Hence:

$$\sum_{\tau=0}^{T_\nu-1}\big[g_{S(t+\tau)}(\boldsymbol{q}(t+\tau)) - g_{S(t+\tau)}(\boldsymbol{q}(t))\big] \leq \sum_{\tau=0}^{T_\nu-1}\big(\tau B^2\big) = \frac{1}{2}(T_\nu^2 B^2 - T_\nu B^2).$$

Plug this into (63), we have:

$$\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \leq \|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 + (T_\nu^2 + T_\nu)B^2 - 2\sum_{\tau=0}^{T_\nu-1}\big[g_{S(t+\tau)}(\boldsymbol{\gamma}_V^*) - g_{S(t+\tau)}(\boldsymbol{q}(t))\big]. \quad (65)$$

Now denote $\mathcal{Z}(t) = (\mathcal{H}(t), \boldsymbol{q}(t))$, i.e., the pair of the history up to time t, $\mathcal{H}(t) = \{S(\tau)\}_{\tau=0}^{t-1}$ and the current backlog. Taking expectations on both sides of (65), conditioning on $\mathcal{Z}(t)$, we have:

$$\begin{aligned}\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} &\leq \mathbb{E}\big\{\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} + (T_\nu^2 + T_\nu)B^2 \\ &\quad -2\mathbb{E}\Big\{\sum_{\tau=0}^{T_\nu-1}\big[g_{S(t+\tau)}(\boldsymbol{\gamma}_V^*) - g_{S(t+\tau)}(\boldsymbol{q}(t))\big] \mid \mathcal{Z}(t)\Big\}.\end{aligned}$$

Since the number of times $g_{s_i}(\gamma)$ appears in the interval $[t, t+T_\nu-1]$ is $\|\mathcal{T}_{s_i}(t, T_\nu)\|$, we can rewrite the above as:

$$\begin{aligned}\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} &\leq \mathbb{E}\big\{\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} + (T_\nu^2 + T_\nu)B^2 \\ &\quad -2T_\nu\mathbb{E}\Big\{\sum_{i=1}^{M}\frac{\|\mathcal{T}_{s_i}(t, T_\nu)\|}{T_\nu}\big[g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big] \mid \mathcal{Z}(t)\Big\}.\end{aligned}$$

Adding and subtracting $2T_\nu\sum_{i=1}^M p_{s_i}\big[g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big]$ from the right hand side, we have:

$$\begin{aligned}\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} &\leq \mathbb{E}\big\{\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} + (T_\nu^2 + T_\nu)B^2 \\ &\quad -2T_\nu\sum_{i=1}^{M}p_{s_i}\big[g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big] \\ &\quad -2T_\nu\mathbb{E}\Big\{\sum_{i=1}^{M}\Big[\frac{\|\mathcal{T}_{s_i}(t, T_\nu)\|}{T_\nu} - p_{s_i}\Big]\times \\ &\qquad \big[g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big] \mid \mathcal{Z}(t)\Big\}.\end{aligned} \quad (66)$$

Denote the last term of (66) as $\mathcal{Q}$, and using the fact that $g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))$ is a constant given $\mathcal{Z}(t)$, we have:

$$\begin{aligned}\mathcal{Q} &= -2T_\nu\sum_{i=1}^{M}\Big[\frac{\mathbb{E}\big\{\|\mathcal{T}_{s_i}(t, T_\nu)\| \mid \mathcal{Z}(t)\big\}}{T_\nu} - p_{s_i}\Big] \times\big[g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big] \\ &\leq 2T_\nu\sum_{i=1}^{M}\Big|\frac{\mathbb{E}\big\{\|\mathcal{T}_{s_i}(t, T_\nu)\| \mid \mathcal{Z}(t)\big\}}{T_\nu} - p_{s_i}\Big| \times\big|g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big|\end{aligned}$$

By (59), $g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t)) \leq B\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\|$, thus we have:

$$\begin{aligned}\mathcal{Q} &\leq 2T_\nu B\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\|\sum_{i=1}^{M}\Big|\frac{\mathbb{E}\big\{\|\mathcal{T}_{s_i}(t, T_\nu)\| \mid \mathcal{Z}(t)\big\}}{T_\nu} - p_{s_i}\Big| \\ &\leq 2T_\nu\nu B\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\|,\end{aligned} \quad (67)$$

where the last step follows from the definition of T_ν. Now by (11) and (58):

$$\sum_{i=1}^{M}p_{s_i}\big[g_{s_i}(\boldsymbol{\gamma}_V^*) - g_{s_i}(\boldsymbol{q}(t))\big] = g(\boldsymbol{\gamma}_V^*) - g(\boldsymbol{q}(t)).$$

Plug this and (67) into (66), we have:

$$\begin{aligned}\mathbb{E}\big\{\|\boldsymbol{q}(t+T_\nu) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} &\leq \mathbb{E}\big\{\|\boldsymbol{q}(t) - \boldsymbol{\gamma}_V^*\|^2 \mid \mathcal{Z}(t)\big\} + (T_\nu^2 + T_\nu)B^2 \\ &\quad -2T_\nu\big(g(\boldsymbol{\gamma}_V^*) - g(\boldsymbol{q}(t))\big) + 2T_\nu\nu B\|\boldsymbol{\gamma}_V^* - \boldsymbol{q}(t)\|.\end{aligned}$$

Recall that $\mathcal{Z}(t) = (\mathcal{H}(t), \boldsymbol{q}(t))$. Taking expectation over $\mathcal{H}(t)$ on both sides proves the lemma. ■

APPENDIX B – PROOF OF (28)

Here we prove that for $\tilde{Y}(t)$ defined in the proof of part (b) of Theorem 1, we have for all $\tilde{Y}(t) \geq 0$ that:

$$\Delta_{T_\nu}(\tilde{Y}(t)) \leq e^{2wT_\nu B} - \frac{w\eta}{2}e^{w\tilde{Y}(t)}.$$

Proof: If $\tilde{Y}(t) > T_\nu B$, denote $\delta(t) = \tilde{Y}(t+T_\nu) - \tilde{Y}(t)$. It is easy to see that $|\delta(t)| \leq T_\nu B$. Rewrite (27) as:

$$\Delta_{T_\nu}(\tilde{Y}(t)) = e^{w\tilde{Y}(t)}\mathbb{E}\big\{\big(e^{w\delta(t)} - 1\big) \mid \boldsymbol{q}(t)\big\}. \quad (68)$$

By a Taylor expansion, we have that:

$$e^{w\delta(t)} = 1 + w\delta(t) + \frac{w^2\delta^2(t)}{2}l(w\delta(t)), \quad (69)$$

where $l(y) = 2\sum_{k=2}^{\infty}\frac{y^{k-2}}{k!} = \frac{2(e^y-1-y)}{y^2}$ [22] has the following properties:

1) $l(0) = 1$; $l(y) \leq 1$ for $y < 0$; $l(y)$ is monotone increasing for $y \geq 0$;
2) For $y < 3$, $l(y) = 2\sum_{k=2}^{\infty}\frac{y^{k-2}}{k!} \leq \sum_{k=2}^{\infty}\frac{y^{k-2}}{3^{k-2}} = \frac{1}{1-y/3}$.

Thus by (69) we have:

$$e^{w\delta(t)} \leq 1 + w\delta(t) + \frac{w^2T_\nu^2B^2}{2}l(wT_\nu B). \quad (70)$$

Plug this into (68), and note that $\tilde{Y}(t) > T_\nu B$, so by (26) we have $\mathbb{E}\big\{\delta(t) \mid \boldsymbol{q}(t)\big\} \leq -\eta$. Hence:

$$\Delta_{T_\nu}(\tilde{Y}(t)) \leq e^{w\tilde{Y}(t)}\big(-w\eta + \frac{w^2T_\nu^2B^2}{2}l(wT_\nu B)\big). \quad (71)$$

Choosing $w = \frac{\eta}{T_\nu^2B^2 + T_\nu B\eta/3}$, we see that $wT_\nu B < 3$, thus:

$$\frac{w^2T_\nu^2B^2}{2}l(wT_\nu B) \leq \frac{w^2T_\nu^2B^2}{2}\frac{1}{1-wT_\nu B/3} = \frac{w\eta}{2},$$

where the last equality follows since:

$$\begin{aligned} w = \frac{\eta}{T_\nu^2B^2 + T_\nu B\eta/3} &\Rightarrow w(T_\nu^2B^2 + T_\nu B\eta/3) = \eta \\ &\Rightarrow wT_\nu^2B^2 = \eta - wT_\nu B\eta/3 \\ &\Rightarrow wT_\nu^2B^2\frac{1}{1-wT_\nu B/3} = \eta. \end{aligned}$$

Therefore (71) becomes:

$$\Delta_{T_\nu}(\tilde{Y}(t)) \leq -\frac{w\eta}{2}e^{w\tilde{Y}(t)} \leq e^{2wT_\nu B} - \frac{w\eta}{2}e^{w\tilde{Y}(t)}. \quad (72)$$

Now if $\tilde{Y}(t) \leq T_\nu B$, it is easy to see that $\Delta_{T_\nu}(\tilde{Y}(t)) \leq e^{2wT_\nu B} - e^{w\tilde{Y}(t)} \leq e^{2wT_\nu B} - \frac{w\eta}{2}e^{w\tilde{Y}(t)}$, since $\tilde{Y}(t+T_\nu) \leq T_\nu B + \tilde{Y}(t) \leq 2T_\nu B$ and $\frac{w\eta}{2} \leq 1$, as $\eta < T_\nu B$. Therefore for all $\tilde{Y}(t) \geq 0$, we see that (28) holds. ■

APPENDIX C-PROOF OF LEMMA 3

Here we prove Lemma 3. To save space, we will sometimes use $[a]^+$ to denote $\max[a, 0]$.

Proof: It suffices to show that (46) holds for a single queue j. Also, when $\mathcal{W}_j = 0$, (46) trivially holds, thus we only consider $\mathcal{W}_j > 0$.

Part (A): We first prove $q_j(t) \leq \max[W_j(t) - \mathcal{W}_j, 0] + \delta_{max}$. First we see that it holds at $t = 0$, since $W_j(0) = \mathcal{W}_j$ and $q_j(t) = 0$. It also holds for $t = 1$. Since $q_j(0) = 0$ and $W_j(0) = \mathcal{W}_j$, we have $q_j(1) = A_j(0) \leq \delta_{max}$. Thus we have $q_j(1) \leq \max[W_j(1) - \mathcal{W}_j, 0] + \delta_{max}$.

Now assume $q_j(t) \leq \max[W_j(t) - \mathcal{W}_j, 0] + \delta_{max}$ holds for $t = 0, 1, 2, ..., k$, we want to show that it also holds for $t = k+1$. Note that if $q_j(k) \leq \mu_j(k)$, the the result holds since then $q_j(k+1) = [q_j(k) - \mu_j(k)]^+ + A_j(k) = A_j(k) \leq \delta_{max}$. Thus we will consider $q_j(k) \geq \mu_j(k)$ in the following:

(A-I) Suppose $W_j(k) \geq \mathcal{W}_j$. Note that in this case we have:

$$q_j(k) \leq W_j(k) - \mathcal{W}_j + \delta_{max}. \quad (73)$$

Also, $q_j(t+1) = \max[q_j(t) - \mu_j(t), 0] + A_j(t)$. Since $q_j(k) \geq \mu_j(k)$, we have:

$$\begin{aligned} q_j(k+1) &= q_j(k) - \mu_j(k) + A_j(k) \\ &\leq W_j(k) - \mathcal{W}_j + \delta_{max} - \mu_j(k) + A_j(k) \\ &\leq [W_j(k) - \mu_j(k) + A_j(k) - \mathcal{W}_j]^+ + \delta_{max} \\ &\leq \big[[W_j(k) - \mu_j(k)]^+ + A_j(k) - \mathcal{W}_j\big]^+ + \delta_{max} \\ &= \max[W_j(k+1) - \mathcal{W}_j, 0] + \delta_{max}, \end{aligned}$$

where the first inequality is due to (73), the second and third inequalities are due to the $[a]^+$ operator, and the last equality follows from the definition of $W_j(k+1)$.

(A-II) Now suppose $W_j(k) < \mathcal{W}_j$. In this case we have $q_j(k) \leq \delta_{max}$, $\tilde{A}_j(k) = [A_j(k) - \mathcal{W}_j + W_j(k)]^+$ and:

$$q_j(k+1) = [q_j(k) - \mu_j(k)]^+ + \tilde{A}_j(k).$$

First consider the case when $W_j(k) < \mathcal{W}_j - A_j(k)$. In this case $\tilde{A}_j(k) = 0$, so we have:

$$q_j(k+1) = q_j(k) - \mu_j(k) \leq \delta_{max} - \mu_j(k) \leq \delta_{max},$$

which implies $q_j(k+1) \leq \max[W_j(k+1) - \mathcal{W}_j, 0] + \delta_{max}$. Else if $\mathcal{W}_j - A_j(k) \leq W_j(k) < \mathcal{W}_j$, we have:

$$\begin{aligned} q_j(k+1) &= q_j(k) - \mu_j(k) + A_j(k) - \mathcal{W}_j + W_j(k) \\ &\leq W_j(k) - \mathcal{W}_j + \delta_{max} - \mu_j(k) + A_j(k) \\ &\leq \max[W_j(k+1) - \mathcal{W}_j, 0] + \delta_{max}, \end{aligned}$$

where the first inequality uses $q_j(k) \leq \delta_{max}$ and the second inequality follows as in (A-I).

Part (B): We now show that $q_j(t) \geq \max[W_j(t) - \mathcal{W}_j, 0]$. First we see that it holds for $t = 0$ since $W_j(0) = \mathcal{W}_j$. We also have for $t = 1$ that:

$$\begin{aligned} [W_j(1) - \mathcal{W}_j]^+ &= \big[[W_j(0) - \mu_j(0)]^+ + A_j(0) - \mathcal{W}_j\big]^+ \\ &\leq \big[[W_j(0) - \mu_j(0) - \mathcal{W}_j]^+ + A_j(0)\big]^+ \\ &= A_j(0) \end{aligned}$$

Thus $q_j(1) \geq \max[W_j(1) - \mathcal{W}_j, 0]$ since $q_j(1) = A_j(0)$. Now suppose $q_j(t) \geq \max[W_j(t) - \mathcal{W}_j, 0]$ holds for $t = 0, 1, ..., k$, we will show that it holds for $t = k+1$. We note that if $W_j(k+1) < \mathcal{W}_j$, then $\max[W_j(k+1) - \mathcal{W}_j, 0] = 0$ and we are done. So we consider $W_j(k+1) \geq \mathcal{W}_j$.

(B-I) First if $W_j(k) \geq \mathcal{W}_j$, we have $\tilde{A}_j(k) = A_j(k)$. Hence:

$$\begin{aligned} [W_j(k+1) - \mathcal{W}_j]^+ &= [W_j(k) - \mu_j(k)]^+ + A_j(k) - \mathcal{W}_j \\ &\leq [W_j(k) - \mu_j(k) - \mathcal{W}_j]^+ + A_j(k) \\ &\leq [[W_j(k) - \mathcal{W}_j]^+ - \mu_j(k)]^+ + A_j(k) \\ &\leq [q_j(k) - \mu_j(k)]^+ + A_j(k), \end{aligned}$$

where the first two inequalities are due to the $[a]^+$ operator and the last inequality is due to $q_j(k) \geq [W_j(k) - \mathcal{W}_j]^+$. This implies $[W_j(k+1) - \mathcal{W}_j]^+ \leq q_j(k+1)$.

(B-II) Suppose $W_j(k) < \mathcal{W}_j$. Since $W_j(k+1) \geq \mathcal{W}_j$, we have $\mathcal{W}_j - A_j(k) \leq W_j(k) < \mathcal{W}_j$, for otherwise $W_j(k) < \mathcal{W}_j - A_j(k)$ and $W_j(k+1) = [W_j(k) - \mu_j(t)]^+ + A_j(t) < \mathcal{W}_j$. Hence in this case $\tilde{A}_j(k) = A_j(k) - \mathcal{W}_j + W_j(k) \geq 0$.

$$\begin{aligned} & [W_j(k+1) - \mathcal{W}_j]^+ \\ = \;& [W_j(k) - \mu_j(k)]^+ + A_j(k) - \mathcal{W}_j \\ \leq \;& [W_j(k) + q_j(k) - \mu_j(k)]^+ + A_j(k) - \mathcal{W}_j \\ \leq \;& [q_j(k) - \mu_j(k)]^+ + A_j(k) - \mathcal{W}_j + W_j(k) \\ = \;& q_j(k+1), \end{aligned}$$

where the two inequalities are due to the fact that $q_j(k) \geq 0$ and $W_j(k) \geq 0$. ■

REFERENCES

[1] Y. Yi and M. Chiang. Stochastic network utility maximization: A tribute to kelly's paper published in this journal a decade ago. *European Transactions on Telecommunications*, vol. 19, no. 4, pp. 421-442, June 2008.

[2] L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Trans. on Automatic Control, vol. 37, no. 12, pp. 1936-1949*, Dec. 1992.

[3] M. J. Neely. Energy optimal control for time-varying wireless networks. *IEEE Transactions on Information Theory 52(7): 2915-2934*, July 2006.

[4] M. J. Neely, E. Modiano, and C. Li. Fairness and optimal stochastic control for heterogeneous networks. *IEEE INFOCOM Proceedings*, March 2005.

[5] L. Huang and M. J. Neely. The optimality of two prices: Maximizing revenue in a stochastic network. *Proc. of 45th Annual Allerton Conference on Communication, Control, and Computing (invited paper)*, Sept. 2007.

[6] R. Urgaonkar and M. J. Neely. Opportunistic scheduling with reliability guarantees in cognitive radio networks. *IEEE INFOCOM Proceedings*, April 2008.

[7] L. Georgiadis, M. J. Neely, and L. Tassiulas. *Resource Allocation and Cross-Layer Control in Wireless Networks*. Foundations and Trends in Networking Vol. 1, no. 1, pp. 1-144, 2006.

[8] M. J. Neely. Optimal energy and delay tradeoffs for multi-user wireless downlinks. *IEEE Transactions on Information Theory vol. 53, no. 9, pp. 3095-3113*, Sept. 2007.

[9] M. J. Neely. Super-fast delay tradeoffs for utility optimal fair scheduling in wireless networks. *IEEE Journal on Selected Areas in Communications (JSAC), Special Issue on Nonlinear Optimization of Communication Systems*, 24(8), Aug. 2006.

[10] A. Eryilmaz and R. Srikant. Fair resource allocation in wireless networks using queue-length-based scheduling and congestion control. *IEEE/ACM Trans. Netw.*, 15(6):1333–1344, 2007.

[11] M. J. Neely and R. Urgaonkar. Opportunism, backpressure, and stochastic optimization with the wireless broadcast advantage. *Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA (Invited paper)*, Oct. 2008.

[12] S. Moeller, A. Sridharan, B. Krishnamachari, and O. Gnawali. Routing without routes: The backpressure collection protocol. *9th ACM/IEEE International Conference on Information Processing in Sensor Networks (IPSN)*, 2010.

[13] M. J. Neely. *Dynamic Power Allocation and Routing for Satellite and Wireless Networks with Time Varying Channels*. PhD thesis, Massachusetts Institute of Technology, Laboratory for Information and Decision Systems (LIDS), 2003.

[14] M. J. Neely. Stability and capacity regions or discrete time queueing networks. *arXiv:1003.3396v1*, March 2010.

[15] A. L. Stolyar. Greedy primal-dual algorithm for dynamic resource allocation in complex networks. *Queueing Systems*, Vol. 54, No.3, pp.203-220, 2006.

[16] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar. *Convex Analysis and Optimization*. Boston: Athena Scientific, 2003.

[17] M. J. Neely, E. Modiano, and C. E. Rohrs. Dynamic power allocation and routing for time-varying wireless networks. *IEEE Journal on Selected Areas in Communications, Vol 23, NO.1*, January 2005.

[18] L. Huang and M. J. Neely. Delay reduction via lagrange multipliers in stochastic network optimization. *arXiv:0904.3795v1*, April 2009.

[19] M. J. Neely. Intelligent packet dropping for optimal energy-delay tradeoffs in wireless downlinks. *Proc. of the 4th Int. Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt)*, April 2006.

[20] F. Kelly. Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, vol. 8, pp. 33-37, 1997.

[21] C. Curescu and S. Nadjm-Tehrani. Price/utility-based optimized resource allocation in wireless ad hoc networks. *IEEE SECON*, 85- 95, 2005.

[22] F. Chung and L. Lu. Concentration inequalities and martingale inequalities a survey. *Internet Math.*, 3 (2006-2007), 79–127.